Positivity of direct images and algebraicity of holomorphic foliations on compact Kähler manifolds

(joint with J. Cao)

Simons Collaboration on
Moduli of Varieties
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1. Introduction

2. Positivity of direct images

3. Pseudo-effectivity of line and vector bundles

4. Algebraicity criteria for foliations

5. Proof of the main theorems

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- So far, there is no *differential geometric* argument.

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- In case $\mathcal{F} = T_X$, this is due to W. Ou.
- It follows that X is uniruled and K_X not pseff.
- Main techniques involved in the proofs: positivity of direct images and algebraicity criteria for holomorphic foliations

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Let $p : X \to Y$ be a holomorphic surjective map between Kähler manifolds. (L, h_L) is a line bundle over X, such that $\sqrt{-1}\Theta_{h_L}(L) \geq 0$. Assume that the multiplier ideal sheaf of h_L is trivial, and that $(K_X + L)|_{X_y}$ is pseff.

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  - For $m \geq 0$, consider $\mathcal{G}_m := \pi_\star(K_{\mathbb{P}(E^\star)/X} \otimes \mathcal{O}_{\mathbb{P}(E^\star)}(m+r+1) \otimes \mathcal{I}(mh_\epsilon))$. It is a non-trivial subsheaf of $\mathrm{Sym}^m(E) \otimes \det E$.
  - Curvature: $\sqrt{-1}\Theta_{h_{\mathcal{G}_m}}(\mathcal{G}_m) \geq -C\omega_X \otimes \mathrm{Id}$. Compute the degree, take $m \to \infty$.

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  - Other applications of this method: numerical characterisation of the Kähler cone, positivity of direct images...

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- By considering the relative MRC fibration for p, one shows that X_y are rationally connected.

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