Kähler-Ricci flow and MMP

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Moduli of Varieties

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Setup

X^n compact Kähler manifold

$H^{1,1}(X,\mathbb{R}) \subset H^2(X,\mathbb{R})$ de Rham classes of closed real $(1,1)$-forms

Kähler cone

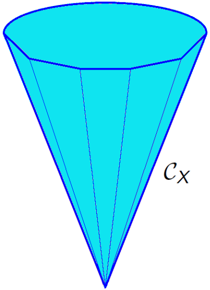
$$\mathcal{C}_X = \{[\omega] \in H^{1,1}(X,\mathbb{R}) \mid \omega \text{ Kähler metric on } X\}$$

Open convex cone in $H^{1,1}(X,\mathbb{R})$

$\overline{\mathcal{C}_X}$ =nef cone

ω Kähler metric, Ricci curvature form $\mathrm{Ric}(\omega) \overset{loc}{=} -\frac{i}{2\pi}\partial\overline{\partial} \log \det g$

$[\mathrm{Ric}(\omega)] = -c_1(K_X) \in H^{1,1}(X,\mathbb{R})$



Kähler-Ricci flow

(X^n, ω_0) compact Kähler manifold. A smooth family of Kähler metric $\omega(t)$ on X, $t \in [0, T)$ solves the Kähler-Ricci flow if

$$\begin{cases} \frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)) \\ \omega(0) = \omega_0 \end{cases}$$

Theorem (Hamilton 82)

(X^n, ω_0) compact Kähler manifold. The Kähler-Ricci flow has a unique solution for some maximal time interval, $t \in [0, T)$, $0 < T \leqslant \infty$.

In cohomology we have $\frac{\partial}{\partial t}[\omega(t)] = c_1(K_X)$, hence $[\omega(t)] = [\omega_0] + tc_1(K_X) \in H^{1,1}(X, \mathbb{R})$

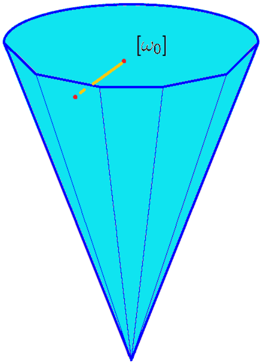
Theorem (Tian-Zhang 06)

The maximal existence time of the Kähler-Ricci flow is

$$T = \sup\{t > 0 \mid [\omega_0] + tc_1(K_X) \in \mathcal{C}_X\} \leqslant \infty$$

Maximal existence time

$$T = \sup\{t > 0 \mid [\omega_0] + tc_1(K_X) \in \mathcal{C}_X\} \leqslant \infty$$



Corollary

$T = \infty \Leftrightarrow c_1(K_X) \in \overline{\mathcal{C}_X}$, *i.e.* K_X *is nef*

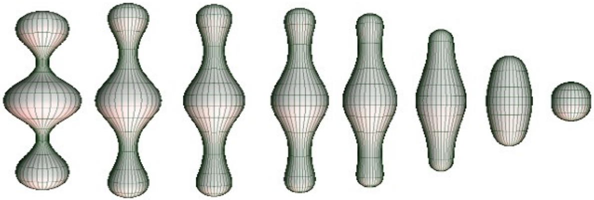
2006: Cascini-La Nave and Song-Tian proposed a direct relation between MMP (with scaling) and the behavior of the KRF (when X projective and $[\omega_0] = c_1(H)$)

We will illustrate this by studying the behavior of the flow. Ultimate goal: understand the behavior of $\omega(t), t \to T$, in full generality.

Curves

C smooth curve (Hamilton 88, Chow 91)

- $K_C \cong \mathcal{O}_C$: $T = \infty$, $\omega(t)$ converges smoothly to a Ricci-flat metric
- K_C ample: $T = \infty$, $\omega(t)/t$ converges smoothly to a negative Kähler-Einstein metric, $\mathrm{Ric}(\omega_{\mathrm{KE}}) = -\omega_{\mathrm{KE}}$
- $-K_C$ ample: $T < \infty$, $\mathrm{Vol}(X, \omega(t)) \to 0$, $\omega(t)/(T-t)$ converges smoothly to a positive Kähler-Einstein metric, $\mathrm{Ric}(\omega_{\mathrm{FS}}) = \omega_{\mathrm{FS}}$



Surfaces

S compact Kähler surface

Assume K_S not nef, so $T < \infty$.

• If $\mathrm{Vol}(S, \omega(t)) \geqslant c > 0$ (i.e. if $[\omega_0] + TK_S$ nef and big), then $\pi : S \to S'$ blowup of finitely many disjoint (-1)-curves and $[\omega_0] + TK_S = \pi^*[\omega_{S'}]$.

Song-Weinkove 10: $\omega(t) \to \pi^*\omega_T$ smoothly away from $\mathrm{Exc}(\pi)$. Flow can be restarted on S' and the process is continuous in the Gromov-Hausdorff topology

• Suppose $\mathrm{Vol}(S, \omega(t)) \to 0$ (i.e. $[\omega_0] + TK_S$ not big).

∘ If $[\omega_0] + TK_S$ nontrivial class, then $\pi : S \to C$ a $\mathbb{P}^1$-bundle and $[\omega_0] + TK_S = \pi^*[\omega_C]$.

Expect $\omega(t) \to \pi^*\omega_C$ smoothly. Unknown even when $S = \mathbb{P}^1 \times \mathbb{P}^1$.

∘ If $[\omega_0] + TK_S = 0$ then S Fano. Then $\omega(t)/(T - t)$ converges smoothly to a Kähler-Ricci soliton, $\mathrm{Ric}(\omega_{\mathrm{KRS}}) = \omega_{\mathrm{KRS}} + L_V\omega_{\mathrm{KRS}}$ (work of many many people)

Surfaces

S compact Kähler surface

Assume K_S nef, so $T = \infty$.

• $Kod(S) = 2$: S minimal of general type, $\omega(t)/t$ converges smoothly away from $\mathbb{B}_+(K_S)$ to a negative Kähler-Einstein metric, pullback of orbifold KE metric on the canonical model $S \to S_{\mathrm{can}}$ (Tian-Zhang 06)

• $Kod(S) = 1$: $\pi : S \to C$ minimal properly elliptic surface, $\omega(t)/t$ converges smoothly away from singular fibers of π to a twisted Kähler-Einstein metric on C:
$\mathrm{Ric}(\omega_{\mathrm{TKE}}) = -\omega_{\mathrm{TKE}} + \omega_{\mathrm{WP}}$ (Song-Tian 06, Gross-T.-Zhang 11, Fong-Zhang 12)

• $Kod(S) = 0$: $K_S \sim_{\mathbb{Q}} \mathcal{O}_S$, $\omega(t)$ converges smoothly to a Ricci-flat metric (Cao 85)

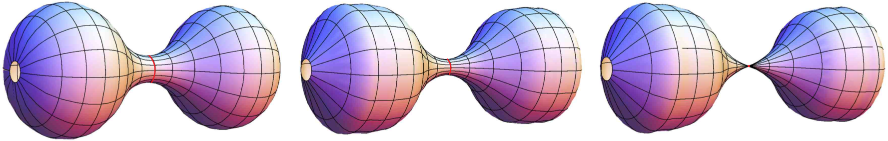
Finite-time singularities

(X^n, ω_0) compact Kähler manifold, $\omega(t)$: solution of KRF starting at ω_0

Assume K_S not nef, so $T < \infty$. Call $[\alpha] = [\omega_0] + TK_X$ limiting nef class.

Singularity formation set $\Sigma \subset X$ defined by

$$X \backslash \Sigma = \{x \in X \mid \text{curvature of } \omega(t) \text{ is locally bounded near } x \text{ as } t \to T\}$$



Conjecture (Feldman-Ilmanen-Knopf 03)

The singularity formation set Σ is a closed analytic subvariety of X.

Finite-time singularities

Theorem (Collins-T. 13)

The Conjecture is true, and more precisely

$$\Sigma = \bigcup_{\int_V \alpha^{\dim V}=0} V =: \mathrm{Null}([\alpha]),$$

is the union of all subvarieties V *with* $\mathrm{Vol}(V, \omega(t)) \to 0$ *as* $t \to T$. *Away from* Σ, *the metrics* $\omega(t)$ *converge smoothly to a limiting Kähler metric* ω_T.

Expect that Σ can be contracted:

Conjecture (Filip-T. 17 – Transcendental BPF)

X *compact Kähler manifold,* $[\alpha] \in H^{1,1}(X, \mathbb{R})$ *nef with* $[\alpha] - c_1(K_X)$ *nef and big. Then there is a contraction* $f : X \to Y$ *with* $[\alpha] = f^*[\omega_Y]$.

True if $n \leqslant 3$ (Filip-T. 17, T.-Zhang 18, Höring 18, Das-Hacon 20), and for all n if X projective (Das-Hacon 25).

Finite-time singularities

Conjecture (Filip-T. 17 – Transcendental BPF)

X compact Kähler manifold, $[\alpha] \in H^{1,1}(X, \mathbb{R})$ nef with $[\alpha] - c_1(K_X)$ nef and big. Then there is a contraction $f : X \to Y$ with $[\alpha] = f^*[\omega_Y]$.

Apply this to a finite-time singularity of KRF, with $[\alpha] = [\omega_0] + TK_X$.

If $[\alpha]$ is not big, then f is a Fano fibration ($-K_X$ is f-ample). Expect: flow collapses the fibers, $\omega(t) \to f^*\omega_T$ away from the singular fibers, and $\omega(t)/(T-t)|_{\mathrm{fiber}}$ converges weakly to a Kähler-Ricci soliton. Known when $Y = \mathrm{pt}$ (Perelman, Bamler, Chen-Wang,...), only very partial results otherwise.

If $[\alpha]$ is big, then f is bimeromorphic, and either a divisorial or flipping contraction. Assuming flips exist, expect flow can be restarted on the new variety, and the process is continuous in the Gromov-Hausdorff topology. Partial results by Song-Tian 09 in the projective case.

Diameter behavior

Conjecture

At any finite-time singularity of KRF we have $\mathrm{diam}(X, \omega(t)) \leqslant C$.

Theorem (Guo-Phong-Song-Sturm 23)

This is true when $[\alpha]$ *is big.*

Conjecture (Tian 08)

At a finite-time singularity of KRF we have "extinction" (i.e. $\mathrm{diam}(X, \omega(t)) \to 0$ *as* $t \to T$*) if and only if* X *Fano and* $[\omega_0] = -Tc_1(K_X)$.

After partial results by Song 14, T.-Y.Zhang 18,

Theorem (J.Zhang 25)

This is true.

Immortal solutions

Expect: after finitely many singularities (either volume non-collapsed or volume collapsed), the flow either becomes extinct, or it exists for all time.

From now on assume (X^n, ω_0) compact Kähler manifold with K_X nef, to $T = +\infty$.

Without assuming Abundance, we have

Theorem (Guo-Phong-Song-Sturm 23)

We have $\mathrm{diam}(X, \omega(t)/t) \leqslant C$*, and we can take sequential Gromov-Hausdorff limits* $(X, \omega(t_i)/t_i) \to (Z, d)$ *compact metric space.*

Expect (Z, d) independent of sequence, and homeomorphic to a projective variety, the canonical model of X.

Weak limit

Conjecture (T. 24)

(X^n, ω_0) compact Kähler manifold with K_X nef, $\omega(t)$ KRF starting at ω_0. Then $\omega(t)/t$ converges weakly to a closed positive current $0 \leqslant \eta \in c_1(K_X)$ with "minimal singularities", independent of ω_0.

η would be the curvature of a "canonical" semipositively curved singular metric on K_X.

Expect also η to be smooth on a Zariski open subset. The kernel of η should define a foliation on X, whose leaves should be closed and give the Iitaka fibration on X.

All of this is true when K_X is semiample, as we will see.

Immortal solutions

If we assume Abundance, much more is known.

X^n compact Kähler manifold with K_X semiample.

• $Kod(X) = n$: X minimal of general type, $\omega(t)/t$ converges smoothly away from $\mathbb{B}_+(K_X)$ to a negative Kähler-Einstein metric, pullback of singular KE metric on the canonical model $X \to X_{\mathrm{can}}$ (Tian-Zhang 06, Eyssidieux-Guedj-Zeriahi 06)

• $Kod(X) = 0$: $K_X \sim_{\mathbb{Q}} \mathcal{O}_X$, $\omega(t)$ converges smoothly to a Ricci-flat metric (Cao 85)

• $0 < Kod(X) < n$: $\pi : X \to Y$ Iitaka fibration, general fibers are Calabi-Yau. Away from $D = \mathrm{disc}(\pi) \subset Y$ there is a Weil-Petersson form $\omega_{\mathrm{WP}} \geqslant 0$, and Song-Tian 08 constructed a twisted Kähler-Einstein metric on $Y \backslash D$: $\mathrm{Ric}(\omega_{\mathrm{TKE}}) = -\omega_{\mathrm{TKE}} + \omega_{\mathrm{WP}}$. They proved $\frac{\omega(t)}{t} \to \pi^*\omega_{\mathrm{TKE}}$ weakly.

Smooth collapsing

Conjecture (Song-Tian 08)

In this case we have $\frac{\omega(t)}{t} \to \pi^*\omega_{\mathrm{TKE}}$ smoothly on $X\backslash\pi^{-1}(D)$, and with locally uniformly bounded Ricci curvature.

After many partial results (Fong-Zhang 12, Gill 13, T.-Weinkove-Yang 14, T.-Zhang 14, Fong-Lee 20, Chu-Lee 21,...) finally

Theorem (Hein-Lee-T. 25)

The conjecture is true.

Also, $\omega(t)|_{\pi^{-1}(x)}$ converges smoothly to a Ricci-flat metric on any smooth Calabi-Yau fiber.

Any sequential Gromov-Hausdorff limit of $(X, \omega(t)/t)$ is homeomorphic to Y (Lee-T.-Zhang 26). Expect that it is isometric to the metric completion of $(Y\backslash D, \omega_{\mathrm{TKE}})$. Known when Y smooth and D snc divisor (Li-T. 23)

Outlook

So far, all the progress has gone in the direction MMP $\rightsquigarrow$ KRF

If Abundance is proved, we would have a basically complete understanding of immortal solutions of the flow

But if MMP is proved, there is still a lot of work to do to understand finite time singularities of the flow

Challenge: find something useful in the other direction KRF $\rightsquigarrow$ MMP

Perhaps some monotone quantity along the flow (e.g. Perelman entropy) can be used?

Limit of the flow reminescent of Thurston-Perelman Geometrization of 3-manifolds

Thank You !