Recent progress in the Kaehler minimal model program

Christopher Hacon

University of Utah

June, 2025

MMP conjecture

- We would like to generalize the results from the mmp for projective varieties to the context of Kähler varieties.

Conjecture (Kähler BCHM)

Let (X, ω) be a compact Kähler variety, (X, B) be a klt pair.

1. *If $K_X + B$ is big or $K_X + B$ is pseudo-effective and B is big, then (X, B) has a good minimal model.*
2. *If $K_X + B$ is not pseudo-effective, then (X, B) has a Mori fiber space.*

- We begin by recalling the various definitions in the analytic context.

Positivity

- Since divisors can be rare on Kähler varieties, it is convenient to define nef, big, pseff, Kähler, etc. for classes $\alpha \in H^{1,1}_{\rm BC}(X)$. This is the space generated by locally $\partial\bar\partial$ exact (1,1) forms on X (think of these as the analog of $\mathbb{R}$-Cartier divisors).
- ω is a Kähler form if it is > 0. $\alpha \in H^{1,1}_{\rm BC}(X)$ is Kähler if $\int_Y \alpha^k \wedge \omega^{\dim Y - k} > 0$ for any subvariety $Y \subset X$.
- The nef cone is the closure of the Kähler cone so $\int_Y \alpha^k \wedge \omega^{\dim Y - k} \geq 0$ for any subvariety $Y \subset X$.
- Note that even K_X is not a globally defined divisor. It is a reflexive rank 1 sheaf and if it is $\mathbb{Q}$-Cartier, then $c_1(K_X) \in H^{1,1}_{\rm BC}(X)$.
- $\alpha \in H^{1,1}_{\rm BC}(X)$ is big/pseudo-effective if it is represented by a Kähler/positive current.

Positivity

- We let $N^1(X) = H^{1,1}_{\mathrm{BC}}(X)$ and $N_1(X) = (N^1(X))^\vee$.
- $\rho(X) := \dim H^{1,1}_{\mathrm{BC}}(X)$.
- $\mathcal{K} \subset H^{1,1}_{\mathrm{BC}}(X)$ is the Kähler cone and $\overline{\mathcal{K}} \subset H^{1,1}_{\mathrm{BC}}(X)$ is the nef cone
- Since curves are rare on Kähler varieties, it is convenient to think of $\mathrm{NA}(X) = \mathcal{K}^\vee \subset N_1(X)$ the cone of positive classes generated by $\int_Y \omega^{\dim Y - 1} \wedge \cdots$.
- We have $\mathrm{NE}(X) = < \int_C \ldots > \subset \mathrm{NA}(X)$.

Generalized pairs

- It is convenient to work with generalized pairs $(X, B+\boldsymbol{\beta})$.
- Traditionally $\boldsymbol{\beta}$ is a nef b-$\mathbb{R}$-divisor. For us $\boldsymbol{\beta}$ is a nef b-(1,1)-form.
- This means that for a resolution $\nu : X' \to X$, $\boldsymbol{\beta}_{X'} \in H^{1,1}_{\rm BC}(X')$ is nef and if $p : W \to X'$, $q : W \to X''$, then $\boldsymbol{\beta}_{X''} = q_*(p^*\boldsymbol{\beta}_{X'})$.
- By definition of gen pair,

$$\nu_*(K_{X'} + B' + \boldsymbol{\beta}_{X'}) = K_X + B + \boldsymbol{\beta}_X \in H^{1,1}_{\rm BC}(X)$$

  where $\boldsymbol{\beta}$ is b-nef and descends to X'.
- gklt/glc if B' snc and the coefficients of B' are $< 1/\leq 1$.
- Define $a_E(X, B, \boldsymbol{\beta}) = -{\rm mult}_E(B')$.

Minimal model/Mori fiber spaces

- We say that $(X, B+\boldsymbol{\beta})$ is a minimal model if $K_X + B + \boldsymbol{\beta}_X$ is nef.
- $(X, B+\boldsymbol{\beta})$ is a good minimal model if there exists a morphism $f : X \to Z$ and a Kähler form ω_Z s.t. $K_X + B + \boldsymbol{\beta}_X \equiv f^*\omega_Z$
- $(X, B+\boldsymbol{\beta}) \to Z$ is a Mori fiber space if $\rho(X/Z) = 1$, $\dim X > \dim Z$, $-(K_X + B + \boldsymbol{\beta}_X)$ is relatively Kähler.

Conjecture (Generalized Kähler BCHM)

Let (X, ω) be a compact Kähler variety, $(X, B+\boldsymbol{\beta})$ be a gklt pair.

1. *If $K_X + B + \boldsymbol{\beta}$ is big or $K_X + B + \boldsymbol{\beta}$ is pseudo-effective and $B+\boldsymbol{\beta}$ is big, then $(X, B+\boldsymbol{\beta})$ has a good minimal model $X \dashrightarrow X'$.*
2. *If $K_X + B + \boldsymbol{\beta}$ is not pseudo-effective, then $(X, B+\boldsymbol{\beta})$ has a Mori fiber space $X \dashrightarrow X'$.*

- $X \dashrightarrow X'$ birational cont. $a_E(X, B+\boldsymbol{\beta}) \leq a_E(X, B'+\boldsymbol{\beta})$

Projective MMP

- Notice that the Kähler mmp is more general than [BCHM] even for projective varieties. (However, the traditional case $\beta = \mathrm{N}$ follows from [BCHM].)

Theorem (Hacon-Das)

Let X be a projective variety and $(X, B + \beta)$ a gklt pair, then **Kähler BCHM** *holds for $(X, B + \beta)$.*

- The key observation is that if X is projective, then $\beta' \equiv N' + \delta$ where N' is a nef $\mathbb{R}$-divisor and $\int_C \delta = 0$ for any curve $C \subset X'$.
- In order to establish the **g-Kähler BCHM** conjecture and to run the MMP we will need the cone theorem, the contraction theorem, the existence of flips, and the termination of flips.

Cone Theorem

Theorem (Hacon-Paun)

Let $(X, B+\beta)$ be a gklt pair with $B+\beta_X$ big, then

$$\mathrm{NA}(X) = \mathrm{NA}(X)_{(K_X+B+\beta_X)\geq 0} + \sum_{i\in I} \mathbb{R}^+[\Gamma_i]$$

where $\{\Gamma_i\}$ is a finite set of rational curves with

$$0 < -(K_X+B+\beta_X)\cdot\Gamma_i \leq 2\dim X.$$

- If X contains no rational curves then $K_X+B+\beta_X$ is nef.
- I will discuss the proof later.

Existence of flips

Theorem (Das-Hacon)

Let $(X, B + \beta_X)$ be a gklt pair, $f : X \to Z$ be a flipping contraction, then the flip $f^+ : X^+ \to Z$ exists.

- Note $\rho(X/Z) = \rho(X^+/Z) = 1$, $-(K_X + B + \beta_X)$ and $K_{X^+} + B^+ + \beta_{X^+}$ are Kähler over Z. Note that f is birational, hence Moishezon and we have

Theorem (Fujino, Das-Hacon-Paun)

The results of [BCHM] hold for projective morphisms.

- This theorem holds for traditional generalized pairs $(X, B + \mathrm{N})$ and we reduce to this case.

Termination of flips

- Termination of flips for klt projective pairs is known to be a very difficult problem in dimension ≥ 4.
- Termination of flips for g-klt pairs is even harder (eg. β_X is not NQC).
- So the idea is to run a minimal model program with scaling $(X, B+\beta_X+t\omega)$ and then we obtain a sequence of flips and div contractions $X \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \ldots$
- We have a sequence $\lambda_1 > \lambda_2 > \ldots \geq 0$ such that $X \dashrightarrow X_i$ is a min model for $(X, B+\beta_X+t\omega_X)$ for $\lambda_{i-1} \geq t \geq \lambda_i$.
- We then use finiteness of minimal models (when $B+\beta_X$ is big) to show termination.
- This works in dim $X=3$ (Das-Hacon-Yanez) and hopefully in all dimensions.

Transcendental base point free conjecture

Conjecture (Transcendental base point free conjecture)

If $(X, B+\beta)$ is gklt and $\alpha = K_X + B + \beta_X$ is nef and big and generalized klt, then α is semiample i.e. there is a morphism $f : X \to Z$ such that $\alpha \equiv f^*\alpha_Z$ where α_Z is Kähler on Z.

- As mentioned above this holds for projective varieties.
- We can reduce to the case where (X, B) is klt and β_X is Kähler, and this is the traditional setting of the Kähler Ricci flow.
- For completeness we mention the following.

Conjecture (Abundance)

If $(X, B+\beta)$ is klt and $K_X + B + \beta_X$ is nef, then it is semiample.

Current Status

- Everything works for projective pairs (Fujino, Das-Hacon-Paun, Das-Hacon).
- Everything works in dimension 3 (Campana, Horing, Peternell, Das, Hacon, Ou, Yanez).
- There is substantial progress in dimension 4:
  [Das-Hacon-Paun 2023] for (X, B) klt 4-fold such that $K_X + B \equiv D \geq 0$ and
  [Das-Hacon in progress] "weak mmp" for (X, B) klt 4-fold.
- There is a clear approach in higher dimension (but many technical issues.....so....)
- In what follows I will describe the main ingredients for a proof of the Cone Theorem and an approach to the Transcendental Base Point Free Conjecture.

Ou's result + applications

Theorem (BDPP Conjecture [Ou 2025])

Let X be a Kähler variety, then X is uniruled iff K_X is not pseudo-effective.

- In our setting $B + \beta_X$ is big so if $K_X + B + \beta_X$ is not big, then K_X is not pseudo-effective.
- We then have a non-trivial MRC $f : X \dashrightarrow Z$.
- By [Claudon-Horing 2024] f is Moishezon.
- Replacing X by a resolution, we may assume that f is a projective morphism and we run a relative mmp $X \dashrightarrow X' \ /Z$.
- If $K_X + B + \beta_X$ is not pseff over Z, then this ends with a $(K_X + B + \beta_X)$-MFS $g : X' \to Z'$.

Ou's result + applications

- If $K_X + B + \beta_X$ is pseff over Z, then this ends with a $(K_X + B + \beta_X)$-trivial fibration $g : X' \to Z'$.
- By the canonical bundle formula [Hacon-Paun 2024] we have $K_{X'} + B' + \beta_{X'} = g^*(K_{Z'} + B_{Z'} + \gamma_{Z'})$ where $(Z', B_{Z'} + \gamma)$ is a generalized pair and $B_{Z'} + \gamma_{Z'}$ is big.
- We conclude by induction on the dimension.

Theorem

If $(X, B + \beta)$ is gklt and $K_X + B + \beta_X$ is not pseff, then there is a "MFS" $X \dashrightarrow X' \to Z$.

- So the interesting case is when $K_X + B + \beta_X$ is big.
- We focus on $\alpha = K_X + B + \beta_X$ big and nef but not Kähler (eg running the mmp with scaling).
- For simplicity consider $\alpha = K_X + B + \omega$ where ω is Kähler.

Null loci

Theorem (Collins-Tosatti, Hacon-Paun)

If α is nef and big, then $\mathrm{Null}(\alpha) := \cup\{V | \int_V \alpha^d = 0\}$ is a closed analytic subset with no isolated points and there is a representative $\gamma \equiv \alpha$ with nice sings and $E_{nK}(\gamma) = \mathrm{Null}(\alpha)$.

- The idea is that if $\gamma \in [\alpha]$ is a Kähler current with minimal singularities, then $E_{nK}(\gamma) \supset \mathrm{Null}(\alpha)$.
- If V is a max component of $E_{nK}(\gamma)$ not contained in $\mathrm{Null}(\alpha)$, then $\alpha|_V$ is big (as $\int_V \alpha^d > 0$) and so there is a (1,1) form $\psi_V \equiv \alpha|_V$ with $E_{nK}(\psi_V)$ strictly contained in V.
- We then extend ψ_V to ψ_U on a neighborhood U of V and define γ' by taking the max of ψ_U and γ near V.
- The resulting $\gamma' \equiv \gamma$ has strictly smaller non-Kähler locus.

On the cone theorem

Theorem (Cao-Horing, Hacon-Paun)

If (X, ω) is Kähler, $(X, B + \boldsymbol{\beta})$ is gklt such that $\alpha = K_X + B + \boldsymbol{\beta}_X + \omega$ is nef, then there is a α-trivial rational curve C such that $0 > (K_X + B + \boldsymbol{\beta}_X) \cdot C = \omega \cdot C \geq -2 \dim X$.

- C then generates a negative extremal ray......
- The idea is as follows.
- If α is not big, then K_X is not pseff and so we apply Ou's result on the BDPP Conjecture.

On the cone theorem

- If α is big, then we pick $\gamma \equiv \alpha$ such that $E_{nK}(\gamma) = \mathrm{Null}(\alpha)$.
- We pick a top dimen'l component V of $E_{nK}(\gamma)$, and λ the log canonical threshold of $(X, B+\beta; \gamma)$ at general points of V.
- Thus $(X, B+\beta+\lambda\gamma)$ is properly log canonical at general points of V.
- By a subadjunction result of Hacon-Paun, we have $B_V \geq 0$, γ_V nef b-(1,1) form s.t. $B_V + \gamma_V$ big and

$$((1+\lambda)\alpha)|_V = (K_X + B + \beta + \lambda\gamma)|_V = K_V + B_V + \gamma_V$$

- Thus K_V is not pseudo-effective and by the Ou's result on the BDPP Conjecture, V is covered by rational curves □

- Next we illustrate an approach to the Transcendental BPF Conjecture (**Highly speculative and incomplete!**).
- Suppose that $(X, B+\beta)$ is gklt, $\alpha = K_X + B + \beta_X$ is nef and $B+\beta_X$ is big. We aim to define $g : X \to Z$ such that $\alpha = g^*\omega_Z$ where ω_Z is Kähler.
- If α is not big, then K_X is not PSEF and we consider the MRC $X \dashrightarrow Z$. As mentioned above, we plan to proceed by induction on the dimension.
- If α is big then $N = \mathrm{Null}(\alpha) = E_{nK}(\gamma)$ is an analytic subset where $\gamma \equiv \alpha$ is a Kähler current.
- Since α is Kähler outside N, then the corresponding contraction $g : X \to Z$ should be the identity on $X \setminus N$.
- Our approach is divided in 3 steps.

On the transcendental BPF conjecture

- **Step 1.** Define $f|_N : N \to W$.
- We let λ_i be the jumping numbers for $(X, B + \boldsymbol{\beta} + t\gamma)$ and $V_i = \mathrm{Nklt}(X, B + \boldsymbol{\beta}_X + \lambda_i\gamma)$ (with reduced structure).
- $(1+\lambda_1)\alpha|_{V_1} = (K_X + B + \boldsymbol{\beta} + \lambda_1\gamma)|_{V_1} = K_{V_1} + B_{V_1} + \eta_{V_1}$ is a gklt pair, η_{V_1} is big and so $\alpha|_{V_1}$ is semi-ample by induction on the dimension.
- In particular $f_1 : V_1 \to W_1$ is defined.
- Suppose that $f_k : V_k \to W_k$ is defined (contracting α-trivial curves), and $V_{k+1} = V_k \cup V'$.
- $(1+\lambda_{k+1})\alpha|_{V'} \equiv (K_X + B + \boldsymbol{\beta} + \lambda_{k+1}\gamma)|_{V'} = K_{V'} + B_{V'} + \eta_{V'}$ is a g-pair and $\mathrm{nklt}(V', B_{V'} + \eta_{V'}) \subset V_k$ so that $\alpha|_{V' \cap V_k}$ is semi-ample.
- By induction on the dimension we obtain $f' : V' \to W'$ and gluing this along $V' \cap V_k$ we get $f_{k+1} : V_{k+1} \to W_{k+1}$.

- **Step 2.** Replacing $(X, B+\beta_X)$ by a log resolution, we may assume that $N = \mathrm{Null}(\alpha) = \mathrm{Nklt}(K_X + B + \beta_X + \lambda\eta)$ for $\lambda \gg 0$ is a divisor.
- Assume for simplicity that $B \geq 0$, β_X is nef, $\gamma = F + \omega$ where ω Kähler, F snc div supported on N.
- We will define $\tilde{f}_k : \tilde{V}_k \to \tilde{W}_k$ where $\tilde{V}_k = \lfloor B + \lambda_k F \rfloor$.
- We get short exact sequences

$$0 \to \mathcal{O}_{V'}(-\lfloor B + \lambda_k F \rfloor) \to \mathcal{O}_{\tilde{V}_{k+1}} \to \mathcal{O}_{\tilde{V}_k} \to 0$$

- Since $\gamma \equiv_f 0$, we have $(-\lfloor B + \lambda_k F \rfloor)|_{V'} \equiv_f$ $(K_X + V' + \{B + \lambda_k F\} + \beta_X + \lambda_k \omega)|_{V'} = K_{V'} + B_{V'} + \beta_{V'}$
- So $f|_{V'}$ is proj, and by Kawamata-Viehweg vanishing $R^1 f_* \mathcal{O}_{V'}(-\lfloor B + \lambda_k F \rfloor) = 0$
- Thus $f_*\mathcal{O}_{\tilde{V}_{k+1}} \to f_*\mathcal{O}_{\tilde{V}_k}$ is surjective.
- Thus f extends to arbitrary thickenings of N.

On the transcendental BPF conjecture

- **Step 3.** Extending f to X.
- Since $\gamma = F + \omega$ is f-trivial, $-F|_{\mathrm{Supp} F}$ is relatively ample.
- For $k \gg 0$ sufficiently divisible, we will have that
  1. $A := kF$ is a Cartier divisor,
  2. f extends to A,
  3. $-A|_A$ is ample, and
  4. $R^i f_* \mathcal{O}_A(-jA) = 0$ for all $i > 0$, $j > 0$.
- By [Artin, Fujiki] f extends to $g : X \to Z$ where $g|_{X \setminus A} = \mathrm{id}_{X \setminus A}$ and $g|_A = f$.
- This approach works in dim 3, but several technical difficulties must be dealt with in dim ≥ 4.