Uniruled compact Kähler manifolds

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Kähler manifolds

- A Kähler manifold is a complex manifold X endowed with a Kähler form ω, *i.e.* a $\mathcal{C}^\infty$ closed positive real $(1,1)$-form.
- The projective space $\mathbb{P}^N$ is a Kähler manifold. There is a standard Kähler form on it, which is called the Fubini-Study form.
- Every submanifold of a Kähler manifold is a Kähler manifold. More generally, we can define Kähler varieties (after Grauert 1965). They are possibly singular complex analytic varieties endowed with a Kähler form. In particular, every projective variety is Kähler.
- There are compact Kähler manifolds X, such that any bimeromorphic model of X cannot be approximated by projective manifolds (Voisin 2004).

Kähler manifolds

- Every compact Kähler manifold X of dimension at most 3 can be approximated by projective manifolds (Kodaira 1960, Lin 2017).
- Log Minimal Model Programs and Abundance Theorem hold for compact Kähler varieties of dimension at most 3 (Höring-Peternell 2015, Campana, Das, Hacon, Ou...).
- Relative Minimal Model Programs hold for projective morphisms between complex analytic varieties (Das-Hacon-Păun 2022, Fujino 2022).

Rational curves in compact Kähler manifolds

- A rational curve is a compact complex analytic variety whose normalization is isomorphic to $\mathbb{P}^1$.
- Mori's bend-and-break theorem (1979) shows that, if the canonical divisor K_X of a projective manifold X is not nef, then X contains a rational curve C such that $C \cdot K_X < 0$. The proof relies on the reduction modulo p.
- X is called uniruled if it is covered by rational curves. Miyaoka-Mori (1986) showed that X is uniruled if and only if X is covered by curves C with $C \cdot K_X < 0$.
- Boucksom-Demailly-Păun-Peternell (2004) showed that the latter condition is equivalent to that K_X is not pseudoeffective.
- For compact Kähler surfaces, the previous characterization holds (Yau 1974).
  Brunella (2006) proved the characterization for compact Kähler threefolds.

Rational curves in compact Kähler manifolds

- Recently, we proved that a compact Kähler manifold X is uniruled if and only if its canonical line bundle ω_X is not pseudoeffective.
- Combined with previous works, we now know that if ω_X is not nef, then X contains a rational curve (Cao-Höring 2020).
  Furthermore, the cone theorem holds for compact Kähler manifolds (Hacon-Păun 2024).
- It remains to prove the contraction theorem for the MMP.
  This contraction conjecture was already raised in the theory of Kähler-Ricci flows in the 2000s (see Song-Tian, Tosatti, etc.)

Outline of the proof

- We follow the method of Brunella. Let X be a compact Kähler manifold such that ω_X is not pseudoeffective. We need to show that X is uniruled. We may assume that it is not projective. By Kodaira's embedding theorem, there is a non zero holomorphic 2-form $\sigma \in H^0(X, \Omega_X^2)$.

- The contraction with σ defines a morphism $T_X \to \Omega_X^1$. Let $\mathcal{F}$ be its kernel. We first note that $\mathcal{F} \neq T_X$ since $\sigma \neq 0$. Moreover, $\mathcal{F} \neq 0$. Otherwise, σ is generically non degenerate and $\sigma^{\wedge \frac{1}{2}\dim X}$ defines a section of ω_X.

- Since X is Kähler, the Hodge theory implies that σ is closed. Hence $\mathcal{F}$ is a foliation on X.
  If $\dim X = 3$, then $\mathcal{F}^*$ is a non pseudoeffective line bundle ($\mathcal{F}^* < 0$). Brunella managed to show that the closure of the leaves of $\mathcal{F}$ are rational curves. In particular, $\mathcal{F}$ is induced by a meromorphic map $X \dashrightarrow Y$, and the 2-form σ comes from Y.

Outline of the proof

- In higher dimension, we will consider some maximal destabilizer $\mathcal{F}'$ of $\mathcal{F}$. Then $\mathcal{F}'$ is a foliation and its dual $\mathcal{F}'^*$ is a non pseudoeffective reflexive coherent sheaf ($\mathcal{F}'^* < 0$).

- Key step. We show that $\mathcal{F}'$ is induced by a meromorphic map $X \dashrightarrow Y$, and the 2-form σ comes from Y.
  Unlike the case of threefolds, we do not know if the fibers are uniruled (It is now known to be true by Cao-Păun).

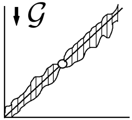
- Since $0 < \dim Y < \dim X$, we can now argue by induction on dimensions and by contradiction as follows. If X is not uniruled, neither is F where F is a general fiber. Thus the canonical line bundle ω_F is pseudoeffective. Hence by the theory of positivity of direct images, $\omega_{X/Y}$ is pseudoeffective.
  It follows that ω_Y is not pseudoeffective and thus Y is uniruled by induction. By considering $X \dashrightarrow Y \dashrightarrow Z$, where $Y \dashrightarrow Z$ is the MRC fibration (rational quotient), we deduce that Y is rationally connected. Hence Y does not have non zero 2-forms. This is a contradiction.

Foliations induced by meromorphic maps

- A foliation $\mathcal{F}$ on a complex manifold is a saturated coherent subsheaf of the tangent bundle T_X, which is closed under the Lie bracket.
- If $f\colon X \to Y$ is a surjective morphism between complex manifolds, then the relative tangent $T_{X/Y}$, which is the kernel of the differential map $\mathrm{d}f\colon T_X \to f^*T_Y$, is a foliation on X.
- A dominant meromorphic map $f\colon X \dashrightarrow Y$ between compact complex manifolds is a morphism $g\colon X' \to Y$, where $X' \to X$ is a composition of blowups. It induces a foliation on X.
- Let X be a compact complex manifold and let $\mathcal{F}$ be a foliation. Let $X^\circ \subseteq X$ be the largest open smooth subset of X where $\mathcal{F}$ is a subbundle of T_X.

Foliations induced by meromorphic maps

- Let $Z = X \times X$ and let $\Delta \subseteq Z$ be the diagonal. There is a foliation $\mathcal{G}$ on Z defined as $p_2^{-1}\mathcal{F} \cap p_1^{-1}0$. Let $Z^\circ = X^\circ \times X^\circ$ and $\Delta^\circ = \Delta \cap Z^\circ$. Then $\mathcal{G}$ is regular on Z° and is transversal to Δ°.
- The analytic (formal) graph Γ° of $\mathcal{F}$ is the union of local leaves of $\mathcal{G}$ passing through points of Δ°. It is a locally closed submanifold of $X \times X$.
  The normal bundle $\mathcal{N}_{\Delta^\circ/S^\circ}$ is isomorphic to $\mathcal{F}|_{X^\circ}$.



Analytic graph

- The foliation $\mathcal{F}$ is induced by a meromorphic map if the Zariski closure of Γ° in $X \times X$ has the same dimension as Γ°.

Zariski closure

- Let X be a compact complex manifold, let S° be an irreducible locally closed submanifold, and let M be the Zariski closure of S°. It is natural to investigate if $\dim M = \dim S^\circ$.
  When the equality holds and when X is projective, we say that S° is algebraic.
- Example: Let $X = \mathbb{P}^2$, $X^\circ = \mathbb{C}^2$ and let $S^\circ \subseteq X^\circ$ be the graph of a holomorphic function φ on $\mathbb{C}$. Then M is a curve if and only if φ is a polynomial, by Chow's theorem.
- Bost's method (Bogomolov-McQuillan, 2001) (Hartshorne 1968). Assume X projective. Let L be an ample line bundle and let $x \in S^\circ$ be a general point.
  For any integer $D, i > 0$, we define the vector subspace

$$E_D^i \subseteq H^0(X, L^{\otimes D})$$

of global sections σ of $L^{\otimes D}$ such that $\sigma|_{S^\circ}$ vanishes at x with order at least i.

Zariski closure

- Then we have $\cdots \supseteq E_D^i \supseteq E_D^{i+1} \supseteq \cdots$. Moreover, $E_D^\infty := \cap_{i \geqslant 1} E_D^i$ is the subspace of global sections σ of $L^{\otimes D}$ which vanishes along S° (hence along M).
- Characterization. S° is algebraic if and only if there is a number $\lambda > 0$ such that $E_D^i = E_D^{i+1} = E_D^\infty$ whenever $i \cdot D^{-1} \geqslant \lambda$.
- Reason. Assume that $\dim M = \dim S$. Let $\sigma \in E_D^i \setminus E_D^\infty$. Then

$$\sigma' := \sigma|_M \in H^0(M, (L|_M)^{\otimes D}).$$

The vanishing order of σ' at x is at least i, for $M = S^\circ$ around x. It follows that $i \cdot D^{-1}$ is bounded from above by some constant depending on x and $L|_M$ (Seshadri constant).

Analytic Viewpoint

- Assume that $0 \neq \sigma \in H^0(X, L^{\otimes D})$. Then σ induces a singular Hermitian metric h on L as follows. Assume that ρ is a local section of L. Then $\frac{\rho^D}{\sigma}$ is a local meromorphic function on X. We then define $h(\rho) = |\frac{\rho^D}{\sigma}|^{\frac{1}{D}}$.

- Let ω be a Kähler form in the class of $c_1(L)$. It follows that σ induces a ω-psh function φ, such that $\varphi = \frac{1}{D}\log|\sigma| + O(1)$.

- Then $\sigma \in E_D^i \setminus E_D^{i+1}$ if and only if the Lelong number satisfies $\nu(\varphi|_{S^\circ}, x) = i \cdot D^{-1}$.

- We expect to adapt Bost's method in the setting of Kähler manifolds, in the language of psh functions and Lelong numbers.

Plurisubharmonic functions and Lelong numbers

- Currents are dual to differential forms with compact supports *via* integration. They are differential forms with distribution coefficients. A subvariety defines a current by taking the integration on it.
- A real locally integrable (L^1_{loc}) function φ on an open domain U of $\mathbb{C}^n$ is called plurisubharmonic (psh) "if" $\mathrm{dd^c}\varphi$ is a positive $(1,1)$-current, and if φ is upper-semicontinuous. Here $\mathrm{dd^c} = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}$.
- Using local charts, we can define psh functions on any complex manifold.
  However, by the maximum principle, any psh function on a compact complex manifold must be constant.
- Let θ be any closed real $\mathcal{C}^\infty$ $(1,1)$-form on a complex manifold X. A real locally integrable upper-semicontinuous function φ on X is called θ-psh if $\mathrm{dd^c}\varphi + \theta$ is a positive current. In general, φ is quasi-psh if it is θ-psh for some θ.

Plurisubharmonic functions and Lelong numbers

- Assume that φ is a quasi-psh function. The Lelong number $\nu(\varphi, x)$ is defined as

$$\nu(\varphi, x) = \sup\{\lambda \geqslant 0 \mid \varphi(y) \leqslant \lambda \log|y-x| + O(1) \text{ around } x\},$$

- φ is said to have analytic singularities around x if locally around x, we can write

$$\varphi = \frac{\alpha}{2} \cdot \log(|g_1|^2 + \cdots + |g_r|^2) + O(1),$$

where $\alpha \geqslant 0$ is a real number, $g_1, ..., g_r$ are holomorphic functions.

- In the situation above, the Lelong number $\nu(\varphi, x)$ is equal to

α multiplied by the minimal vanishing order of $g_1, ..., g_r$ at x.

Algebraic geometry and Kähler geometry

Algebraic setting	Kähler setting
divisors, divisor classes	currents, cohomology classes
$N^1(X)$	$H^{1,1}(X,\mathbb{R})$
ample divisor, ample class	Kähler form, Kähler class
nef class $=$ limit of ample classes	nef class $=$ limit of Kähler classes
psef class $=$ limit of effective classes	psef class $=$ class of a positive current
$A \equiv B$	$\alpha = \beta + \mathrm{dd^c}\varphi$
vanishing order at a point	Lelong number at a point
curve classes in $N_1(X)$	classes in $H^{n-1,n-1}(X,\mathbb{R})$

Analogy of Bost's characterization

Let (X, ω) be a compact Kähler manifold, let $S^\circ \subseteq X$ be an irreducible locally closed submanifold of smaller dimension. Assume that S° is Zariski dense, i.e. $M = X$. Let $\lambda > 0$ be any real number.

Theorem (Simple version)

Let $x \in S^\circ$ be a general point. There is a ω-psh function φ, with analytic singularities, such that $\varphi|_{S^\circ} \not\equiv -\infty$, and that $\nu(\varphi|_{S^\circ}, x) \geqslant \lambda$.

- Bost's criterion says that if S° is algebraic, then there is some $\mu > 0$ such that if $i \cdot D^{-1} \geqslant \mu$, then $i = \infty$.
  Recall that $\sigma \in H^0(X, L^D)$ and i is the vanishing order of $\sigma|_{S^\circ}$ at x.
- Firstly we apply Demailly's mass concentration (1993, relying on Yau's theorem in 1978) to get large $\nu(\varphi|_{S^\circ}, x)$.
  Secondly we apply Demailly's regularization of currents (1992) to get analytic singularities.

Analogy of Bost's characterization

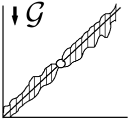
Let (X, ω) be a compact Kähler manifold, let $S^\circ \subseteq X$ be an irreducible locally closed submanifold of smaller dimension. Assume that S° is Zariski dense, i.e. $M = X$. Let $\lambda > 0$ be any real number.

Theorem (General version)

Assume that C is an irreducible compact submanifold of X with $\dim C = \dim S^\circ - 1$. Suppose that S° contains a Zariski open subset C° of C. Furthermore,

1. *the prime divisors on C contained $C \setminus C^\circ$ form an exceptional family,*
2. *S° extends formally along C.*

Then there is a ω-psh function φ, with analytic singularities, such that $\nu(\varphi|_{S^\circ}, x) \geqslant \lambda$ for all $x \in C^\circ$.



Analytic graph

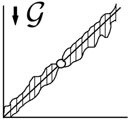
Application

Theorem ("Algebraicity" Criterion I)

Let (X, ω) be a compact Kähler manifold, let $S^\circ \subseteq X$ be an irreducible locally closed submanifold of smaller dimension. Assume that C is an irreducible compact submanifold of X with $\dim C = \dim S^\circ - 1$. Suppose that S° contains a Zariski open subset C° of C. Furthermore,

1. *the prime divisors on C contained $C \setminus C^\circ$ form an exceptional family,*
2. *S° extends formally along C,*
3. *the conormal bundle $\mathcal{N}^*_{C^\circ/S^\circ}$ extends to a line bundle $\mathcal{N}^*$ on C, such that $c_1(\mathcal{N}^*) + \delta$ is not pseudoeffective for any class δ supported in $C \setminus C^\circ$ ($\mathcal{N}^* < 0$).*

Then S° has the same dimension as its Zariski closure.



Analytic graph

Application

Theorem ("Algebraicity" Criterion II)

Let (X, ω) be a compact Kähler manifold, let $S^\circ \subseteq X$ be an irreducible locally closed submanifold of smaller dimension. Assume that C is an irreducible compact submanifold of X. Suppose that S° contains a Zariski open subset C° of C. Furthermore,

1. *the codimension of $C \setminus C^\circ$ is at least 2 in C,*
2. *S° extends formally along C,*
3. *the conormal bundle $\mathcal{N}^*_{C^\circ/S^\circ}$ extends to a reflexive coherent sheaf $\mathcal{N}^*$ on C, which is non pseudoeffective ($\mathcal{N}^* < 0$).*

Then S° has the same dimension as its Zariski closure.

- We blow up C in X, and reduce to the situation of "Algebraicity" Criterion I.

Proof of "Algebraicity" Criterion I

- We assume by contradiction that S° is Zariski dense in X.
- Then by the density theorem, for any given constant λ, there exists a ω-psh function φ, such that $\nu(\varphi|_{S^\circ}, x) = \nu \geqslant \lambda$ for general points $x \in C^\circ$.
- For simplicity, we assume $C = C^\circ$.
- Siu's decomposition.

$$(\omega + \mathrm{dd^c}\varphi)|_{S^\circ} - \nu[C]$$

is a positive current on S°. We can restrict it on C. It follows that

$$\{\omega\}|_C + \nu c_1(\mathcal{N}^*) = \nu \cdot (\frac{1}{\nu}\{\omega\}|_C + c_1(\mathcal{N}^*))$$

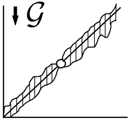
is a pseudoeffective class on C. This is a contradiction for $\mathcal{N}^*$ is not pseudoeffective.

Foliation induced by meromorphic maps

Theorem (Foliation induced by meromorphic maps)

Let (X, ω) be a compact Kähler manifold, let $\mathcal{F}$ be a foliation on X. Assume that $\mathcal{F}^*$ is non pseudoeffective ($\mathcal{F}^* < 0$). Then $\mathcal{F}$ is induced by a meromorphic map.

- Let $\alpha \in H^{n-1,n-1}(X, \mathbb{R})$ be a movable class. If the minimal slope $\mu_{\alpha,min}(\mathcal{F}) > 0$, then $\mathcal{F}^*$ is non pseudoeffective.
- Proof. We consider the the analytic graph of $\mathcal{F}$. It is a locally closed submanifold S° in $Z = X \times X$. It contains $\Delta^\circ \cong X^\circ$, where $\Delta \subseteq Z$ is the diagonal and X° is the regular locus of $\mathcal{F}$. Then the conormal $\mathcal{N}^*_{\Delta^\circ/S^\circ}$ is isomorphic to $\mathcal{F}^*|_{X^\circ}$. We can apply "Algebraicity" Criterion II to conclude.



Analytic graph

Thank you !