On the YTD conjecture for general polarizations

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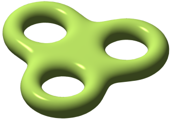
The YTD conjecture

- (X, L) polarized smooth complex projective variety.
- Say "(X, L) cscK" if there exists $\omega \in c_1(L)$ such that

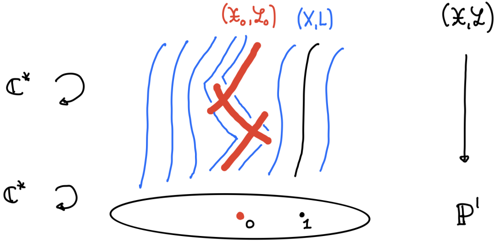
$$\mathrm{Ric}\,\omega \wedge \omega^{n-1} = \frac{\bar{S}}{n}\omega^n.$$



- **Y(TD) conjecture**: (X, L) is cscK iff (X, L) satisfies some AG stability condition.
- Well understood by now when X Fano and $L = -K_X$. Stability is K-polystability.
- We will look at the general polarized case.
- Assume $\mathrm{Aut}(X, L)$ finite for the moment.

Known results

- **Thm** [Boucksom–Hisamoto–J '17]: (X, L) cscK $\implies$ (X, L) uniformly K-stable.
- **Thm** [C. Li '22]: (X, L) uniformly K-stable for models $\implies$ (X, L) cscK.
- *Uniform K-stability* involves (semi)*ample* test configurations $(\mathcal{X}, \mathcal{L})$ for (X, L).



- *Uniform K-stability for models* involves *general* (not necessarily ample) tc's.
- Uniform K-stability for models $\implies$ uniform K-stability.
- **Q**: is the converse true?
- **A**: we don't know!

Main Theorems

- **Thm A** [Boucksom–J '25] If $\mathrm{Aut}(X, L)$ is finite, then

$$(X, L) \text{ cscK} \iff (X, L) \text{ uniformly K-stable for models}$$

- This can be viewed as a positive answer to the YTD conjecture.
- Also have a version when Aut(X, L) is infinite. In fact, have "weighted" version:

    Fix maximal torus $T \subset \mathrm{Aut}(X, L)$.
    Let $P \subset M_{\mathbb{R}}(T)$ be the moment polytope.
    Fix "weight" functions $v, w \in C^\infty(P)$, with $v > 0$.
    Can define the notion of a $(v, w) - cscK$ metric (Lahdili).
    Examples: extremal metrics and Kähler–Ricci solitons.

- **Thm A'** Assume v is log concave. Then

$$(X, L) \text{ is } (v, w)\text{-cscK} \iff (X, L) \text{ is } (v, w)\text{-uniformly K-stable for models}$$

- Will focus on Theorem A to fix ideas. Theorem A' is more technical.

AG and NAG

- Algebro-geometric conditions should only involve (X, L) as an algebraic variety...
- ... but (X, L) canonically induces a Berkovich analytification $(X^{\mathrm{na}}, L^{\mathrm{na}})$ with respect to the *trivial* absolute value on $\mathbb{C}$, leading to *non-Archimedean* geometry.
- Here X^{na} is compact (Hausdorff) and we have dense inclusions

$$X^{\mathrm{na}} \supset X^{\mathrm{val}} \supset X^{\mathrm{divval}}$$

- If $\mathcal{I}$ is the semiring of coherent ideal sheaves on X, then X^{an} is the set of homoms

$$v\colon \mathcal{I} \to [0, +\infty],$$

  and $v \in X^{\mathrm{val}}$ iff $v(I) < +\infty$ for $I \neq 0$.
- Test configurations (ample or not) induce "non-Archimedean" metrics on L^{na}.
- K-stability phrased as conditions on functionals on spaces of metrics on L^{na}.

Overview of proof

The proof uses several steps, not all of which are new (or due to us).

(1) *Coercivity criterion for cscK metrics* [. . . , Chen–Cheng '21,. . .]: reduces the existence of a cscK metric to the behavior (coercivity) of the *Mabuchi functional*.

(2) *Geodesic rays* [Berman–B–J '21]: suffices to have coercivity along geodesic rays.

(3) *Maximal geodesic rays* [BBJ '21]. These are induced by non-Archimedean metrics. By [Li '22], it suffices to consider coercivity along such rays.

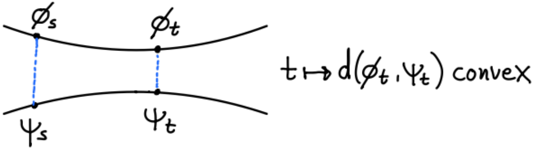
(4) *Asymptotics along maximal geodesic rays*. The slope at infinity of the Mabuchi functional along a maximal geodesic ray is *always* equal to the non-Archimedean Mabuchi functional. *This is the key new step*.

(5) *Non-Archimedean regularization*. The rays obtained in (3) could come from very singular non-Archimedean metrics. By a regularization argument, it is enough to consider metrics associated to general test configurations.

The weighted case presents challenges. . . but is ultimately handled in the same way.

Coercivity criterion for cscK metrics

- $\mathcal{H} := \mathcal{H}(X, L) := \{$smooth positive metrics on $L\}$.
- $\mathcal{E} := \mathcal{E}^1(X, L)$ completion of $\mathcal{H}$ wrt *Darvas metric* d.
- Berman–Darvas–Lu '17: $\mathcal{E}$ is Buseman convex (wrt psh geodesics).



- CscK metrics are critical points of the *Mabuchi functional* $\mathrm{M} \colon \mathcal{H} \to \mathbb{R}$.
- Berman–Darvas–Lu '17: the Mabuchi functional extends to $\mathrm{M} \colon \mathcal{E} \to \mathbb{R} \cup \{+\infty\}$.
- Chen–Cheng '21: (X, L) cscK iff M is *coercive* on $\mathcal{E}$:

$$\mathrm{M} \geq \sigma d(\cdot, \phi_{\mathrm{ref}}) - C \quad \text{where } \sigma, C > 0.$$

- Versions for weighted cscK metrics by He '19, Apostolov–Jubert–Lahdili '23, Di Nezza–Jubert–Lahdili '24-25, Han–Liu '25.

Coercivity and geodesic rays

- $\hat{\mathcal{E}} := \{$geodesic rays $\hat{\phi} = (\phi_t)_{t\geq 0}$ in $\mathcal{E}$ starting at $\phi_{\mathrm{ref}}\}$. This comes with
  - a canonical point $0 \in \hat{\mathcal{E}}$ (the constant ray);
  - a metric $\hat{d}$: $d((\phi_t),(\psi_t)) = \lim_{t\to+\infty} \frac{1}{t} d(\phi_t,\psi_t)$.

- [Berman–Berndtsson '17, BDL '17]: M is convex along psh geodesics.
- $\hat{\mathrm{M}}\colon \hat{\mathcal{E}} \to \mathbb{R}\cup\{+\infty\}$ slope of Mabuchi at infinity.

$$\mathrm{M}(\hat{\phi}) = \lim_{t\to+\infty} \frac{1}{t}\mathrm{M}(\phi_t).$$

- By [BBJ '21, CC '21,...] suffices to test coercivity along geodesic rays:

$$\mathrm{M} \geq \sigma d(\cdot,\phi_{\mathrm{ref}}) - C \text{ on } \mathcal{E} \iff \hat{\mathrm{M}} \geq \sigma \hat{d}(\cdot,0) \text{ on } \hat{\mathcal{E}}$$

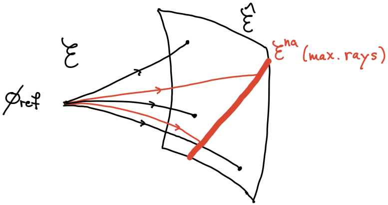
- This is like a Hilbert–Mumford criterion in an infinite-dimensional setting.

Stability in terms of non-Archimedean geometry

- There are non-Archimedean versions of the spaces and functionals above.
- $\mathcal{H}^{\mathrm{na}}$ is the space of *ample* test configurations for (X, L).
- This has a Darvas metric d^{na}, with completion $\mathcal{E}^{\mathrm{na}}$.
- The elements of $\mathcal{E}^{\mathrm{na}}$ can be viewed as (singular) metrics on L^{na}. Examples include metrics coming from *general* (not necessarily ample) test configurations.
- There is a Mabuchi functional $\mathrm{M}^{\mathrm{na}} \colon \mathcal{E}^{\mathrm{na}} \to \mathbb{R} \cup \{+\infty\}$.
- *Uniform K-stability* means $\mathrm{M}^{\mathrm{na}} \geq \sigma d^{\mathrm{na}}(\cdot, 0)$ on $\mathcal{H}^{\mathrm{na}}$, $\sigma > 0$.
- *Uniform K-stability for models* means $\mathrm{M}^{\mathrm{na}} \geq \sigma d^{\mathrm{na}}(\cdot, 0)$ on $\mathcal{E}^{\mathrm{na}}$, $\sigma > 0$.
- Regularization $\Rightarrow$ suffices to check last condition on *general* test configurations.
- Unclear if it suffices to test on $\mathcal{H}^{\mathrm{na}}$ i.e. *ample* test configurations.

Maximal geodesic rays and coercivity

- By [BBJ '21], every non-Archimedean metric induces a geodesic ray in $\mathcal{E}$.
- We get an embedding $\mathcal{E}^{\mathrm{na}} \hookrightarrow \hat{\mathcal{E}}$, which is in fact isometric.



- The rays in the image are called *maximal* geodesic rays.
- The max ray associated to a general test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ corresp. to the largest S^1-invariant psh metric on $\mathcal{L}|_{\mathbb{D}}$ with boundary conditions ϕ_{ref} above $\partial\mathbb{D}$.
- By [Li '21], $\hat{\mathrm{M}}(\hat{\phi}) = +\infty$ for any non-maximal geodesic ray $\hat{\phi}$.
- Thus, suffices to check coercivity along *maximal* geodesic rays.

Mabuchi asymptotics along maximal geodesic rays

- As we can test coercivity along maximal geodesic rays, we in principle have a non-Archimedean (=algebraic!) criterion for (X, L) being cscK.
- To make this more concrete, we need to compute the slope $\hat{\mathrm{M}}$ at infinity of the Mabuchi functional along a maximal geodesic ray.
- Key new result: $\hat{\mathrm{M}} = \mathrm{M}^{\mathrm{na}}$ for *any* maximal geodesic ray.
- This implies the main theorem (when $\mathrm{Aut}(X, L)$ is finite).
- Earlier results:
  - [BHJ '17]: $\hat{\mathrm{M}} = \mathrm{M}^{\mathrm{na}}$ for any maximal geodesic ray coming from an *ample* test configuration. Thus (X, L) cscK $\Rightarrow$ (X, L) uniformly K-stable.
  - [Li '21]: $\mathrm{M}^{\mathrm{na}} \leq \hat{\mathrm{M}}$ for any maximal geodesic ray. Thus (X, L) uniformly K-stable for models $\Rightarrow$ (X, L) cscK.

Asymptotics along maximal rays

- How to control the Mabuchi functional along maximal geodesic rays?
- What is the Mabuchi functional anyway?
- We have $\mathsf{M} = \mathsf{H} + \mathsf{R} + \bar{S}\mathsf{E}$ and $\mathsf{M}^{\mathrm{na}} = \mathsf{H}^{\mathrm{na}} + \mathsf{R}^{\mathrm{na}} + \bar{S}\,\mathsf{E}^{\mathrm{na}}$.
- Here R and E are *energy* terms that are easier to control. The *entropy* term H is more subtle. We have

$$H(\phi) = \int_X \log \frac{\mathsf{MA}(\phi)}{\mathsf{MA}(\phi_{\mathrm{ref}})}\,\mathsf{MA}(\phi) \quad \text{and} \quad \mathsf{H}^{\mathrm{na}}(\varphi) = \int_{X^{\mathrm{na}}} A_X\,\mathrm{MA}^{\mathrm{na}}(\varphi),$$

  - $\mathrm{MA}^{\mathrm{na}}(\phi)$ and $\mathrm{MA}^{\mathrm{na}}(\varphi)$ are *Monge–Ampère measures* on X and X^{na}, resp.;
  - $A_X \colon X^{\mathrm{na}} \to [0, +\infty]$ is the log discrepancy.
- Must show that if $\hat{\phi} = (\phi_t)_{t\geq 0}$ is the geodesic ray in $\mathcal{E}$ starting at ϕ_{ref} and directed by $\varphi \in \mathcal{E}^{\mathrm{na}}$, then $\hat{\mathsf{H}}(\hat{\phi}) := \lim_{t\to\infty} t^{-1} H(\phi_t) = \mathsf{H}^{\mathrm{na}}(\varphi)$.

Regularization and test configurations

- Must show that if $(\phi_t)_{t\geq 0}$ is the geodesic ray in $\mathcal{E}$ starting at ϕ_{ref} and directed by $\varphi \in \mathcal{E}^{\mathrm{na}}$, then $\lim_{t\to\infty} t^{-1}\mathsf{H}(\phi_t) = \mathsf{H}^{\mathrm{na}}(\varphi)$.
- By regularization *may assume* φ *comes from a general test configuration* $(\mathcal{X}, \mathcal{L})$.
- The regularization argument relies on solving a non-Archimedean Monge–Ampère equation [B–Favre–J '15], [BJ '22].
- If φ comes from $(\mathcal{X}, \mathcal{L})$, we have useful formulas. If $\mathcal{X}_0 = \sum_i b_i E_i$, then

$$\mathrm{MA}^{\mathrm{na}}(\varphi) = \sum_i b_i \,\mathrm{vol}_{\mathcal{X}|E_i}(\mathcal{L})\delta_{v_i} \quad \text{and} \quad \mathsf{H}^{\mathrm{na}}(\varphi) = \sum_i b_i \,\mathrm{vol}_{\mathcal{X}|E_i}(\mathcal{L}) A_X(v_i). \qquad (\star)$$

where $v_i = b_i^{-1}\,\mathrm{ord}_{E_i} \in X^{\mathrm{div}}$ corresponds to E_i.

Asymptotics along rays induced by general test configurations

- Let $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ be a general (not nec. ample) test configuration for (X, L).
- There are several asymptotic base loci of $\mathcal{L}$:

$$\mathbb{B}_-^{\mathrm{rel}}(\mathcal{L}) \subset \mathbb{B}_+^{\mathrm{rel}}(\mathcal{L}) \subset \mathbb{B}_+(\mathcal{L}) \subset \mathcal{X}.$$

- After perturbation, we may assume

$$\mathbb{B}_-^{\mathrm{rel}}(\mathcal{L}) = \mathbb{B}_+^{\mathrm{rel}}(\mathcal{L}) = \mathbb{B}_+(\mathcal{L}) \subset \mathcal{X}_0.$$

  In particular, $\mathcal{L}$ is big.
- Pick a smooth, S^1-inv metric Ψ on $\mathcal{L}$, and let $\Phi := \mathrm{P}_{\mathcal{L}}(\Psi)$ be its psh envelope.
- The metric Φ gives rise to a (non-geodesic) ray in $\mathcal{E}$ which approximates the geodesic ray $(\phi_t)_{t\geq 0}$ well. The latter arises from an envelope on $\mathcal{L}|_{\mathbb{D}}$.
- We can now control the entropy along this ray using two facts:
  - [Berman–Demailly '12, Di Nezza–Trapani '24], the singularities of Φ are controlled by $\mathbb{B}_+(\mathcal{L})$.
  - By $(\star)$, $\mathrm{H}^{\mathrm{na}}(\varphi)$ is calculated via $\mathbb{B}_-^{\mathrm{rel}}(\mathcal{L})$.

The weighted version

- The strategy when $\mathrm{Aut}(X, L)$ is infinite is similar.
- Pick $T \subset \mathrm{Aut}(X, L)$ maximal (complex) torus, with compact torus $T_c \subset T(\mathbb{C})$.
- This time, $\mathcal{E}$ consists of T_c-invariant metrics, and T acts on $\mathcal{E}$.
- Coercivity of M now means coercivity on $\mathcal{E}$ wrt the quotient metric on $\mathcal{E}/T$.
- Similarly, $\mathcal{E}^{\mathrm{na}}$ consists of $T(\mathbb{C})$-invariant metrics, and $N_{\mathbb{R}}$ acts on $\mathcal{E}^{\mathrm{na}}$.
- Coercivity on $\mathcal{E}^{\mathrm{na}}$ means wrt the quotient metric on $\mathcal{E}^{\mathrm{na}}/N_{\mathbb{R}}$.
- Added technical difficulties in the weighted case, as we need to develop weighted pluripotential theory (Archimedean and non-Archimedean).
- Extends earlier work was by [Lahdili '19, Han–Li '22, B–J–Trusiani '25. . .]

Comments and questions

- In the Fano case $L = -K_X$ it suffices to consider *special* test configurations. These are associated with (dreamy) divisorial valuations.
- [BJ23]: For (X, L) general, uniform K-stability for models can be expressed as a conditions on *divisorial measures*, convex combinations of divisorial valuations.
- Still unclear if uniform K-stability is equivalent to uniform K-stability for models.
- Uniform K-stability for models is equivalent to the non-uniform inequality $\mathrm{M}^{\mathrm{na}}(\varphi) > 0$ for all $\varphi \in \mathcal{E}^{\mathrm{na}}$. Does it suffice to take φ associated to a test configuration?
- Does the main result hold in the Kähler case? Mesquita-Piccione '24 proved one direction (analogue of Li '22).