Wall crossing for moduli spaces of varieties

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Introduction

1. Existence of moduli spaces of varieties
2. Wall crossing framework
3. Applications

Throughout, we consider projective varieties over $\mathbb{C}$.

KSB moduli spaces

Consider **slc varieties** X with ample canonical divisor K_X and families $\mathcal{X} \to B$ satisfying the Kollár condition.
For example:

- Genus $g \geq 2$ smooth or nodal curves
- Smooth or mildly singular hypersurfaces of degree $d > n + 1$ in $\mathbb{P}^n$
- Surfaces with fixed p_g, q

KSB moduli spaces

Consider **slc varieties** X with ample canonical divisor K_X and families $\mathcal{X} \to B$ satisfying the Kollár condition.

Theorem

For any n, V, there exists a proper Deligne-Mumford stack $\mathcal{M}_{n,V}$ and projective coarse moduli space $M_{n,V}$ parametrizing slc varieties X of dimension n with ample K_X and $(K_X)^n = V$.

For $n > 1$, *very few* complete explicit descriptions of these moduli spaces exist. *Many* partial descriptions are known.

Explicit KSB moduli spaces

Completely described KSB moduli spaces of canonically polarized varieties:

- (Deligne, Mumford) $\overline{\mathscr{M}_g}$: stable genus g curves
- (van Opstall, Liu, Rollenske) surfaces that are quotients and covers of products of curves
- (Alexeev, Pardini) Campedelli and Burniat surfaces

KSB(A) moduli spaces

This has an extension to slc pairs (X, D) with $K_X + D$ ample:

Theorem

For any n, V, and fixed marking of the divisor, there exists a proper Deligne-Mumford stack $\mathcal{M}_{n,V}$ and projective coarse moduli space $M_{n,V}$ parametrizing slc pairs (X, D) such that $\dim X = n$, $K_X + D$ is ample, and $(K_X + D)^n = V$.

A few more (complete) explicit examples are known in this case, especially for $D = \epsilon\Delta$ for $\epsilon \ll 1$ (Alexeev, Hacking, ...). There is a huge body of literature studying explicit KSB(A) moduli spaces.

K-moduli spaces

Consider K-semi/polystable Fano varieties X, i.e. klt varieties X with $-K_X$ ample satisfying an extra stability condition.
For example:

- $\mathbb{P}^n$ is K-polystable
- for (smooth) del Pezzo surfaces:
  - $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, dP6 K-semistable
  - $Bl_p\mathbb{P}^2$, dP7 K-unstable
  - For $d \leq 5$, all are K-stable
- (Chen-Donaldson-Sun, Liu-Xu-Zhuang): A Fano variety X is K-polystable if and only if X admits a KE metric.

K-moduli spaces

Consider K-semi/polystable Fano varieties X.

Theorem

For any n, V, there exists a finite type Artin stack $\mathcal{M}^{ss}_{n,V}$ and projective good moduli space $M^{ps}_{n,V}$ parametrizing K-semi and poly stable varieties X of dimension n with ample $-K_X$ and $(-K_X)^n = V$.

We have complete descriptions of several K-moduli spaces.

Explicit K-moduli spaces

- (Odaka, Spotti, Sun): complete descriptions of K-moduli spaces compactifying the locus of smooth del Pezzo surfaces for each degree
- (Cheltsov, ...): complete descriptions of moduli spaces for many families of Fano threefolds (for which K-moduli is nonempty).

K-moduli spaces

This also has an extension to pairs.

Theorem

For any n, V and fixed rational marking of the divisor, there exists a finite type Artin stack $\mathcal{M}^{ss}_{n,V}$ and projective good moduli space $M^{ps}_{n,V}$ parametrizing K-semi and poly stable pairs (X, D) such that $\dim X = n$, $-(K_X + D)$ is ample, and $(K_X + D)^n = V$.

Prototypical Example

Suppose X is Fano and $D \in |-\frac{1}{r}K_X|$ an ample $\mathbb{Z}$-divisor, $r \in \mathbb{Q}^{>0}$. Then,

$$K_X + cD \text{ is } \begin{array}{l} \text{antiample if } c < r \\ \text{trivial if } c = r \\ \text{ample } c > r. \end{array}$$

Can use the previous results to construct moduli stacks of pairs:

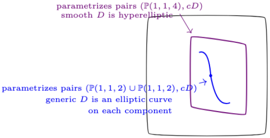
$$\mathcal{M}_c(\mathbb{C}) = \left\{ (Y, cD_Y) \;\middle|\; \begin{array}{l} \dim Y = \dim X \\ (K_Y + cD_Y)^{\dim X} = (K_X + cD)^{\dim X} \\ \text{if } c > r : (X, cD) \text{ slc }, K_X + cD \text{ ample} \\ \text{if } c < r : (X, cD) \text{ K-semistable log Fano} \end{array} \right\}.$$

Question

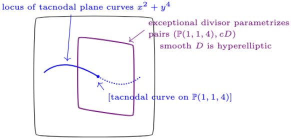
How do these moduli stacks and their associated moduli spaces vary with c? What about $c = r$?

Suppose $X = \mathbb{P}^2$ and D is a degree 4 plane curve.

$\overline{\mathbf{M}}_3 = M_{1-\epsilon}$

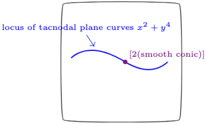


M_c
$5/6 > c > 3/4$



M_c
$3/4 > c > 3/8$

$M_{3/4}$



$M_4^{\mathrm{GIT}} \cong M_c$
$3/8 > c > 0$

Wall crossing

The values of c for which the moduli spaces change are called **walls** and a description of the change in the moduli space from $c_0 + \epsilon$ and $c_0 - \epsilon$ are called **wall crossings**.

There is a general theory of wall crossing for the prototypical example.

Wall crossing

Theorem [Ascher-D-Liu, Zhou]

In the log Fano case, there exists a finite sequence of rational numbers $0 < c_0 < c_1 \cdots < c_k < r$ where the stability conditions change, and

- for $c, c' \in (c_i, c_{i+1}) \cap \mathbb{Q}$, $\mathcal{M}_c \cong \mathcal{M}_{c'}$;
- there are open immersions $\mathcal{M}_{c_i+\epsilon} \hookrightarrow \mathcal{M}_{c_i} \hookleftarrow \mathcal{M}_{c_i-\epsilon}$ of the K-moduli stacks
- the open immersions descend to projective morphisms $M_{c_i+\epsilon} \to M_{c_i} \leftarrow M_{c_i-\epsilon}$ of the K-moduli spaces

This generalizes to the case of multiple divisors and a coefficient vector and recent work of Liu and Zhou extends this to the non-proportional case $D \not\sim -\frac{1}{r}K_X$.

Wall crossing

Theorem [Ascher-Bejleri-Inchoistro-Patakfalvi, Meng-Zhuang]

In the log canonically polarized case, there exists a finite sequence of rational numbers $r < c_0 < c_1 \cdots < c_k < 1$ where the stability conditions change, and

- for $c, c' \in (c_i, c_{i+1})$, $\mathcal{M}_c \cong \mathcal{M}_{c'}$;
- there are morphisms $\mathcal{M}_{c_i+\epsilon} \to \mathcal{M}_{c_i} \leftarrow \mathcal{M}_{c_i-\epsilon}$ of the KSB(A)-moduli stacks
- the morphism descend to projective morphisms $M_{c_i+\epsilon} \to M_{c_i} \leftarrow M_{c_i-\epsilon}$ of the KSB(A)-moduli spaces, where the right arrow is an isomorphism up to normalization

This also generalizes to the case of multiple divisors with a coefficient vector and the non-proportional case.

Wall crossing

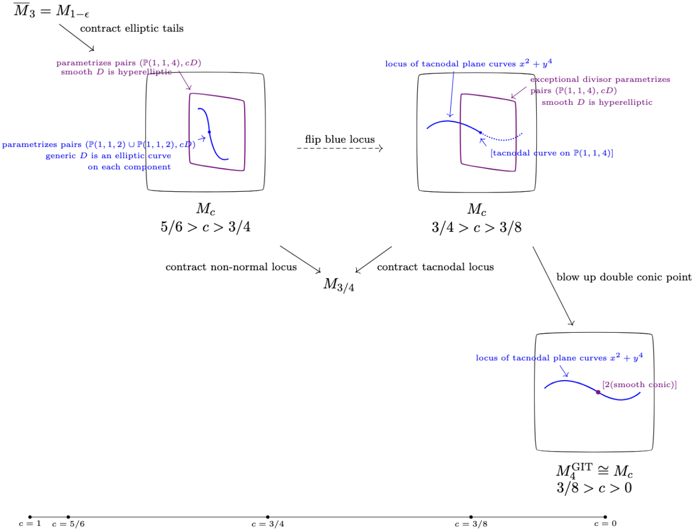
Theorem [Ascher-Bejleri-Blum-D-Inchoistro-Liu-Wang, Blum-Liu]

There is an Artin stack locally of finite type $\mathcal{M}_r$ parametrizing bpCY pairs (X, rD) (slc such that $K_X + rD \sim_{\mathbb{Q}} 0$), whose main component admits an asymptotically good moduli space M_r if $\dim X = 2$, such that

- there are open immersions $\mathcal{M}_{r+\epsilon} \hookrightarrow \mathcal{M}_r \hookleftarrow \mathcal{M}_{r-\epsilon}$ (provided both sides are nonempty);
- the open immersions descend (on the main component) to projective morphisms $M_{r+\epsilon} \to M_r \leftarrow M_{r-\epsilon}$ and M_r is the ample model of the Hodge bundle on either side.

Wall crossing: a typical case

Suppose $X = \mathbb{P}^2$ and D is a degree 4 plane curve.



Applications

Wall crossing gives us an opportunity to interpolate between several different moduli stacks/spaces.

1. The K-moduli spaces tend to be simpler to describe completely
2. For different values of c, these may coincide with other modular compactifications (GIT, BB, ...).
3. We use wall crossing to transport information from one moduli space to the next.

Applications: Chow and Cohomology

Wall crossing for sextic plane curves (Shah, Looijenga, Ascher-D-Liu):

$$\hat{\mathcal{F}}^* = M^K_{c\in(1/4,1/2)}$$

$\mathbb{Q}$−factorialization: $\hat{\mathcal{F}}^* \to \mathcal{F}_2^* = M_{1/2}$

weighted blow up of triple conic: $\hat{\mathcal{F}}^* \to M^K_{c\in(0,1/4)} = M^{\mathrm{GIT}}_{d=6 \text{ plane curves}}$

1. Kirwan-Lee: compute Betti numbers and cohomology of $H^*(\hat{\mathcal{F}}_2^*)$ of Baily-Borel compactification of degree 2 K3 surfaces
2. Canning-Oprea-Pandharipande: compute $A^*(\mathcal{F}_2) = H^{2*}(\mathcal{F}_2)$ the Chow and cohomology ring of the moduli space of degree 2 K3s
3. Ascher-Lee (in progress): compute Chow and cohomology rings of other moduli spaces plane curves.

Idea: use explicit description of wall crossings to determine what classes are added, understand relations from removing classes

Applications: Log canonical models of moduli space

Hassett-Keel Program

Let $\mathbf{M}_g(\alpha) = \mathrm{Proj}\left(R(\overline{\mathscr{M}_g}, K_{\overline{\mathscr{M}_g}} + \alpha\delta)\right)$. Does each variety $\mathbf{M}_g(\alpha)$ have a modular interpretation?

Hassett-Keel-Looijenga Program

Let $\mathcal{F}_{2l}(\mathbf{a}) = \mathrm{Proj}\left(R(\mathcal{F}_{2l}, \lambda + a_1 H_1 + \cdots + a_n H_n)\right)$ for H_i Heegner divisors. Do these have modular interpretations?

Applications: Log canonical models of moduli space

1. Ascher-D-Liu, Hassett: Wall crossing for quartic plane curves recovers the Hassett-Keel program for $\overline{\mathscr{M}_3}$.
2. Ascher-D-Liu-Wang: Complete the Hassett-Keel program in genus 4 (find all walls, provide description of wall crossings)
3. Ascher-D-Liu: Looijenga program for $\mathcal{F}_4^*(a, b)$ degree 4 K3s, interpolating between $\mathcal{F}_4^*$ and GIT of quartic surfaces, verifying conjectures of Laza-O'Grady

Main idea: verify ampleness of appropriate log canonical divisor on each M_c, use uniqueness of log canonical models.

Observation: gives an alternative perspective for Baily-Borel; notably, gives a stack with a universal family whose good moduli space is ($\mathbb{Q}$-factorialization of) $\mathcal{F}^*$ for degree 2 K3s (degree 4 K3s).

Applications: Explicit constructions of K-moduli

K-moduli of Fano threefolds

For a fixed deformation family of smooth Fano threefolds, which are K-(semi/poly)stable? If a general member is K-(semi/poly)stable, can one describe the entire irreducible component of $\mathcal{M}^{ss}(\mathbb{C})$?

Wall crossing input:

- Many of these Fanos occur as covers of other Fanos, e.g. quartic double solids $X =$ double covers of $\mathbb{P}^3$ branched over a quartic S.
  - Liu-Zhu, Zhuang: K-semistability of $X \leftrightarrow$ K-semistability of $(Y, \frac{n-1}{n}D)$
- Others appear as hypersurfaces in Fanos, e.g. quartic hypersurfaces in $\mathbb{P}^n$ for $n \geq 4$.

Leads to studying moduli of pairs (X, cD): start with $c \ll 1$ *('simpler')* and increase coefficient *('more complicated')*.

Applications: Explicit constructions of K-moduli

K-moduli of Fano threefolds

For a fixed deformation family of smooth Fano threefolds, which are K-(semi/poly)stable? If a general member is K-(semi/poly)stable, can one describe the entire irreducible component of $\mathcal{M}^{ss}(\mathbb{C})$?

1. Ascher-D-Liu: Complete description of $\mathcal{M}_{1/2}$ for pairs compactifying locus $(\mathbb{P}^3, \frac{1}{2}S)$ which gives a complete description of the K-moduli of quartic double solids 1-12.
2. Many other examples:
    1. Abban-Cheltsov-Kasprsyk-Liu-Petracci 1-2
    2. Liu-Zhao 2-15
    3. D-Ji-Kennedy-Hunt-Quek 2-18
    4. ...

Applications: Explicit constructions of KSB moduli

KSB moduli of surfaces

Describe the complete KSB moduli spaces for surfaces with fixed numerical invariants.

H- and I- surfaces: double covers of $\mathbb{P}^2$ branched along an octic or $\mathbb{P}(1,1,2)$ branched along a curve of degree 10 $\rightsquigarrow$ studying moduli of pairs $(X, \frac{1}{2}D)$.

In progress (Ascher-D-Liu-Rana-Si):

1. K-moduli wall crossing for pairs (X, cD) for $c \in (0, r)$, $K_X + rD \sim 0$
2. bpCY wall crossing for $c = r$
3. KSBA wall crossing for $c \in (r, \frac{1}{2}) \leftarrow$ in progress
4. Can construct surfaces with *many* components (≥ 30) coming from walls in the $c \in (r, \frac{1}{2})$ region

Applications: Classification

Theorem (Ascher-D-Liu)

The only canonical Gorenstein degenerations of $\mathbb{P}^3$ are $\mathbb{P}^3$, $C(Q, \mathcal{O}_Q(2))$ for Q a smooth quadric surface, $\mathbb{P}(1,1,2,4)$, and X_u.

Method of proof: wall crossing for pairs $(\mathbb{P}^3, cS_4)$.

1. Any such degeneration X must appear in a pair $[(X, (1-\epsilon)D_X)] \in \mathcal{M}_{1-\epsilon}$ for D_X some anticanonical section
2. Study all walls as c increases from 0 to 1; for $c \ll 1$: $\mathcal{M}_c \cong$ GIT of quartic surfaces.
3. Perform explicit computations of replacements of pairs that become unstable to find all $\mathbb{C}$-points of $\mathcal{M}_{1-\epsilon}$.

Can construct other interesting (non-canonical) degenerations of $\mathbb{P}^3$ via wall crossing for $(\mathbb{P}^3, cS_5)$. What about other Fanos?

Applications: complete subvarieties

Wall crossing can also turn hard problems into other hard problems.

Vague question

Let $\mathcal{M}^{sm} \subset \mathcal{M}$ be the locus parametrizing smooth varieties. Do there exist complete subvarieties of $\mathcal{M}^{sm}$?

Precise question

Does there exist a complete subvariety of $\mathscr{M}_g$ of dimension $g-2$? (Open for $g > 3$, Oort's Conjecture.)

Applications: complete subvarieties

- For $g = 3$: wall crossing $\mathbf{M}_3 = M_{1-\epsilon} \dashrightarrow M_\epsilon =$ GIT of quartic plane curves $= \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(4)))//\mathbb{P}GL_3$.
  - Can show: any complete curve in $\mathcal{M}_3$ must map to a complete curve in the GIT moduli space only intersecting the singular locus in the double conic.
  - Find: a curve in $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(4)))$ intersecting the discriminant divisor only in the locus of double conics. What are these curves?
- Similarly for $\mathbf{M}_4$: one can show there exist nontrivial smooth families of $(2, 3)$ complete intersections in $\mathbb{P}^3$ over complete curves and describe families over complete surfaces (if they exist). What are these families?

Thank you!