Arcs & K-stability

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K-stability of Fano varieties has very successful, through:

- moduli theory: there is a projective moduli space of K-polystable Fanos;
- Kähler geometry: K-polystability is equivalent to the existence of a Kähler-Einstein metric.

Questions make sense for general polarised varieties (X, L).

$$\underset{\text{proj}}{X}, \quad \underset{\text{ample}}{L}$$

But we're not even certain of the "right" definition of K-stability.

K-stability is often strengthened to uniform K-stability (a stronger numerical condition) or by including more "test degenerations" (non-Archimedean, arcs).

Goal Give a new <u>geometric</u> interp of a stronger version of K-stability ("uniform K-stability" wrt arcs) and give applications to Kähler geometry.

The interpretation is through Paul's "stability of pairs" (generalisation of GIT). We prove a conjecture around Tian's notion of Υ CM (analytic) stability, and get a new proof of (a version of) YTD for Fanos.

<u>GIT</u>

$X \subset \mathbb{P}(V)$ proj / $\mathbb{C}$, $G \curvearrowright V$ reductive.

Want a "quotient" $X /\!/ G$. GIT does this, but only parametrising "polystable" orbits in X.

<u>Def</u> Let $v \in X$ with <u>lift</u> $\hat{v} \in V$. v is semistable if $\overline{G \cdot \hat{v}} \not\ni 0$.

Thm A (Mumford) There is a quotient $X /\!/ G$ whose points correspond to equivalence classes of semistable orbits, where $v \sim v'$ if $\overline{G.v} \cap \overline{G.v'} \neq \emptyset$.

Defining v to be polystable if $G \cdot \hat{v}$ is closed, the GIT quotient parametrises polystable orbits.

Analytic characterisation: let $|\cdot|$ be a norm on V.

Lemma B (Kempf-Ness) v is semistable $\iff$ $G \to \mathbb{R}$, $g \to \log|g \cdot \hat{v}|$ is bounded below

Numerical char let $\rho : \mathbb{C}^* \to G$ be a one-parameter subgroup and let

$$v_0 = \lim_{t \to 0} \rho(t) \cdot v.$$ Then get

$$\mathbb{C}^* \curvearrowright \langle \hat{v}_0 \rangle \cong \mathbb{C}$$

as $z \to t^{\gamma} z$ for some $\gamma \in \mathbb{Z}$.

Set $\mu(\rho, v) = -\gamma$.

Thm C (Hilbert–Mumford) v is semistable $\iff$ for all 1-PS ρ, we have $\mu(\rho, v) \geq 0$.

Stability of pairs

GIT rarely applies in higher dimensions. Stability of pairs (Paul) gives a flexible generalisation.

Let $G \curvearrowright V, W$ linearly.

Def (Paul) A pair $[v, w] \in \mathbb{P}(V \oplus W)$ is semistable if

$$\overline{G \cdot [v, w]} \cap \mathbb{P}(0 \oplus W) = \emptyset.$$

Ex Let $W = \mathbb{C}$, $w = 1$, trivial action.

Then $\overline{G \cdot [v, 1]} \cap \mathbb{P}(0 \oplus \mathbb{C}) = \emptyset$

$\iff \overline{G \cdot v} \not\ni 0$

$\iff v$ is semistability.

Ex $G \curvearrowright (X, L)$, $\xleftarrow{}$ maybe not ample $L = A - B$ for A, B very ample. Set

$V = H^0(X, A)$, $W = H^0(X, B)$.

We get a viable notion of

semistability for $x \in X$.

Lemma (Paul) $[v,w]$ is semistable
$\iff \exists\ \delta \geq 0$ with

$$\log|g\cdot\hat{v}| - \log|g\cdot\hat{w}| \geq -\delta$$

for all $g \in G$.

"$[v,w]$ is analytically semistable".

Ex (Paul-Sean-Zhang '23) $\exists\ G,\ V,W$ s.t. $[v,w]$ is not semistable but $\mu(\rho,[v,w]) = \mu(\rho,v) - \mu(\rho,w) \geq 0\ \forall \rho$ a 1-PS.

So numerical criterion fails.

Def (Donaldson) An arc is a morphism

$$\operatorname{Spec} \mathbb{C}((t)) \to G$$

$$(\text{or } \Delta^* \to G)$$

So these generalise 1-PS.

For ρ an arc, $\rho\cdot\hat{v}$ and $\rho\cdot\hat{w}$ are vectors of Laurent series.

Def $\mu(\rho,[v,w]) = -\operatorname{ord}_0 \rho\cdot\hat{v} + \operatorname{ord}_0 \rho\cdot\hat{w}$.

Say $[v,w]$ is numerically semistable if $\mu(\rho,[v,w]) \geq 0$ $\forall$ ρ on arc.

<u>Thm</u> (D.-Reboulet) $[v,w]$ is semistable $\iff$ $[v,w]$ is numerically semistable.

We introduce polystable pairs and prove.

<u>Thm</u> (D.-Reboulet) TFAE:

(i) $[v,w]$ is polystable

(ii) $[v,w]$ is numerically polystable

(iii) $[v,w]$ is analytically polystable

applications

<u>applications</u>

Let X smooth proj, dim n, L ample.

$$X \hookrightarrow \mathbb{P}(H^0(X,rL)) = \mathbb{P}^N,$$

defining two points

$R_X \in V$ (Chow point) $\qquad \Delta_X \in W$ (discriminant point)

The set $\{(H_0,\dots,H_{N-n}) \in \mathbb{P}(H^0(X,rL)^*)^{N-n}$
$: X \cap H_0 \cap \dots \cap H_{N-n} \neq \emptyset\}$

is a hypersurface in a Grassmannian
$\rightsquigarrow$ defining section $R_X \in V$.

The set
$$\{(H_0, \dots, H_{N-n-1}) \in \mathbb{P}(H^0(X, rL)^*)^{N-n} :$$
$$\# |X \cap H_0 \cap \dots \cap H_{N-n}| \neq d\}$$
$d = (rL)^n = \deg X$ set $\Delta_X \in W$.

$GL(N+1) \curvearrowright V, W$, so we get a pair $[R_X, \Delta_X] \in \mathbb{P}(V \oplus W)$.

Construction is related to finding $\omega \in c_1(L)$ Kähler with constant scalar curvature, e.g. Kähler-Einstein

The cscK problem is variational: there is a "Mabuchi functional"
$$M : \mathcal{H} \to \mathbb{R}$$
$\uparrow$ space of Kähler metric $c_1(L)$.

s.t. existence of cscK metrics is equivalent (Chen-Cheng) to existence of $\varepsilon, \delta > 0$ s.t.

$\forall \quad \omega \in \mathcal{H}$

$$\mathcal{M}(\omega) \geq \varepsilon \|\omega\|_T - \delta$$

($T \subset \mathrm{Aut}_0(X,L)$ max torus).

Set $\mathcal{H}_r \subset \mathcal{H}$ space of Fubini-Study metrics from $X \subset \mathbb{P}(H^0(X, rL))$

<u>Thm</u> (D.-Reboulet) TFAE:

(i) (X,L) is uniformly K-polystable wrt arcs

(ii) $\exists \; \varepsilon, \delta_r > 0$ s.t.

$$\underbrace{\mathcal{M}(\omega) \geq \varepsilon \|\omega\|_T - \delta_r, \text{ for all } \omega \in \mathcal{H}_r.}$$

"CM polystability" of Tian '97.

Dependence of δ_r on r is all that prevents a solution of YTD.

<u>Cor</u> (D.-Reboulet, Tian, Székelyhidi)

$(X, -K_X)$ Fano ~~man~~ is uniformly K-polystable wrt arcs

$\iff$ $(X, -K_X)$ admits a Kähler-Einstein manifold.

Proofs are applications of stability of pairs. An arc ρ in $GL(N+1) \curvearrowright \mathbb{P}(H^0(X, rL))$ gives a flat family

$$\begin{array}{c}(\mathcal{X}, \mathcal{L}) \\ \downarrow \\ \operatorname{Spec} \mathbb{C}[[t]]\end{array}$$

, general fibre (X, rL), by taking flat limit.

$$DF(\mathcal{X}, \mathcal{L}) = \frac{n}{n+1}\left(\frac{-K_X \cdot L^{n-1}}{L^n}\right)(r^{-1}\mathcal{L})^{n+1} + (r^{-1}\mathcal{L})^n \cdot K_{\mathcal{X}/\operatorname{Spec}}$$

(suitably interpreted), and we prove this (really M^{NA}) is the weight $\mu(\rho, [R_X, \Delta_X])$.

Result of Paul relates M on $\mathcal{H}_r$ to $\log|g \cdot R_X| - \log|g \cdot \Delta_X|$.