Explicit problems on K-stability of Fano varieties

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- K-stability of singular del Pezzo surfaces.
- K-stability of smooth Fano 3-folds.
- K-stability of singular Fano 3-folds.
- K-stability of higher-dimensional Fano varieties.

K-semistable Du Val Del Pezzo surfaces

Classification of Du Val del Pezzo surfaceshas been done Brenton, Bruce, Demazure, Du Val, Gurjar, Hidaka, Hui, Miyanishi, Pradeep, Urabe, Wall, Watanabe, Ye, Zhang.

Theorem (C., Ding, Ghigi, Jeffres, Kollár, Kosta, Liu, Mabuchi, Mukai, Park, Tian, Won)

Let S be a del Pezzo surfaces with at most Du Val singularities. Set $d = (-K_S)^2$. Then S is K-semistable if and only if

- $d = 9$ and S is smooth,
- $d = 8$, S is smooth, and S is not blow up of $\mathbb{P}^2$ at 1 points
- $d = 6$ and S is smooth,
- $d = 5$ and S is smooth,
- $d = 4$ and S has at most $\mathbb{A}_1$ singularities,
- $d = 3$ and S has at most $\mathbb{A}_1$ or $\mathbb{A}_2$ singularities,
- $d = 2$ and S has at most $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$ singularities,
- $d = 1$ and S has at most $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_5$, $\mathbb{A}_6$, $\mathbb{A}_7$, $\mathbb{A}_8$ or $\mathbb{D}_4$ singularities.

δ-invariants of Du Val del Pezzo surfaces (Denisova)

d	# lines	$\mathrm{Sing}(S)$	δ
4	12	$\mathbb{A}_1$	1
4	9	$2\mathbb{A}_1$	1
4	8	$2\mathbb{A}_1$	1
4	6	$3\mathbb{A}_1$	1
4	4	$4\mathbb{A}_1$	1
4	8	$\mathbb{A}_2$	$\frac{6}{7}$
4	6	$\mathbb{A}_2+\mathbb{A}_1$	$\frac{6}{7}$
4	4	$\mathbb{A}_2+2\mathbb{A}_1$	$\frac{6}{7}$

d	# lines	$\mathrm{Sing}(S)$	δ
4	5	$\mathbb{A}_3$	$\frac{3}{4}$
4	4	$\mathbb{A}_3$	$\frac{3}{4}$
4	3	$\mathbb{A}_3+\mathbb{A}_1$	$\frac{3}{4}$
4	2	$\mathbb{A}_3+2\mathbb{A}_1$	$\frac{3}{4}$
4	3	$\mathbb{A}_4$	$\frac{6}{11}$
4	2	$\mathbb{D}_4$	$\frac{1}{2}$
4	1	$\mathbb{D}_5$	$\frac{3}{8}$

δ-invariants of Du Val del Pezzo surfaces (Denisova)

d	# lines	$\mathrm{Sing}(S)$	δ
3	21	$\mathbb{A}_1$	$\frac{6}{5}$
3	16	$2\mathbb{A}_1$	$\frac{6}{5}$
3	12	$3\mathbb{A}_1$	$\frac{6}{5}$
3	9	$4\mathbb{A}_1$	$\frac{6}{5}$
3	15	$\mathbb{A}_2$	1
3	11	$\mathbb{A}_2 + \mathbb{A}_1$	1
3	8	$\mathbb{A}_2 + 2\mathbb{A}_1$	1
3	7	$2\mathbb{A}_2$	1
3	5	$2\mathbb{A}_2 + \mathbb{A}_1$	1
3	3	$3\mathbb{A}_2$	1

d	# lines	$\mathrm{Sing}(S)$	δ
3	10	$\mathbb{A}_3$	$\frac{9}{11}$
3	7	$\mathbb{A}_3 + \mathbb{A}_1$	$\frac{9}{11}$
3	5	$\mathbb{A}_3 + 2\mathbb{A}_1$	$\frac{9}{11}$
3	6	$\mathbb{A}_4$	$\frac{9}{13}$
3	4	$\mathbb{A}_4 + \mathbb{A}_1$	$\frac{9}{13}$
3	3	$\mathbb{A}_5$	$\frac{3}{5}$
3	2	$\mathbb{A}_5 + \mathbb{A}_1$	$\frac{3}{5}$
3	6	$\mathbb{D}_4$	$\frac{3}{5}$
3	3	$\mathbb{D}_5$	$\frac{9}{19}$
3	1	$\mathbb{E}_6$	$\frac{1}{3}$

δ-invariants of Du Val del Pezzo surfaces (Denisova)

d	# lines	$\mathrm{Sing}(S)$	δ
2	44	$\mathbb{A}_1$	$\frac{3}{2}$
2	34	$2\mathbb{A}_1$	$\frac{3}{2}$
2	26	$3\mathbb{A}_1$	$\frac{3}{2}$
2	25	$3\mathbb{A}_1$	$\frac{3}{2}$
2	20	$4\mathbb{A}_1$	$\frac{3}{2}$
2	19	$4\mathbb{A}_1$	$\frac{3}{2}$
2	14	$5\mathbb{A}_1$	$\frac{3}{2}$
2	10	$6\mathbb{A}_1$	$\frac{3}{2}$
2	31	$\mathbb{A}_2$	$\frac{6}{5}$
2	20	$\mathbb{A}_2 + \mathbb{A}_1$	$\frac{6}{5}$
2	18	$\mathbb{A}_2 + 2\mathbb{A}_1$	$\frac{6}{5}$

d	# lines	$\mathrm{Sing}(S)$	δ
2	13	$\mathbb{A}_2 + 3\mathbb{A}_1$	$\frac{6}{5}$
2	16	$2\mathbb{A}_2$	$\frac{6}{5}$
2	12	$2\mathbb{A}_2 + \mathbb{A}_1$	$\frac{6}{5}$
2	8	$3\mathbb{A}_2$	$\frac{6}{5}$
2	22	$\mathbb{A}_3$	1
2	16	$\mathbb{A}_3 + \mathbb{A}_1$	1
2	15	$\mathbb{A}_3 + \mathbb{A}_1$	1
2	12	$\mathbb{A}_3 + 2\mathbb{A}_1$	1
2	11	$\mathbb{A}_3 + 2\mathbb{A}_1$	1
2	8	$\mathbb{A}_3 + 3\mathbb{A}_1$	1
2	10	$\mathbb{A}_3 + \mathbb{A}_2$	1

δ-invariants of Du Val del Pezzo surfaces (Denisova)

d	# lines	$\mathrm{Sing}(S)$	δ
2	7	$\mathbb{A}_3 + \mathbb{A}_2 + \mathbb{A}_1$	1
2	6	$2\mathbb{A}_3$	1
2	4	$2\mathbb{A}_3 + \mathbb{A}_1$	1
2	14	$\mathbb{A}_4$	$\frac{12}{13}$
2	10	$\mathbb{A}_4 + \mathbb{A}_1$	$\frac{12}{13}$
2	6	$\mathbb{A}_4 + \mathbb{A}_2$	$\frac{12}{13}$
2	8	$\mathbb{A}_5$	$\frac{6}{7}$
2	7	$\mathbb{A}_5$	$\frac{3}{4}$
2	6	$\mathbb{A}_5 + \mathbb{A}_1$	$\frac{6}{7}$
2	5	$\mathbb{A}_5 + \mathbb{A}_1$	$\frac{3}{4}$
2	3	$\mathbb{A}_5 + \mathbb{A}_2$	$\frac{3}{4}$
2	4	$\mathbb{A}_6$	$\frac{4}{5}$

d	# lines	$\mathrm{Sing}(S)$	δ
2	2	$\mathbb{A}_7$	$\frac{3}{4}$
2	14	$\mathbb{D}_4$	$\frac{3}{4}$
2	9	$\mathbb{D}_4 + \mathbb{A}_1$	$\frac{3}{4}$
2	6	$\mathbb{D}_4 + 2\mathbb{A}_1$	$\frac{3}{4}$
2	4	$\mathbb{D}_4 + 3\mathbb{A}_1$	$\frac{3}{4}$
2	8	$\mathbb{D}_5$	$\frac{3}{5}$
2	5	$\mathbb{D}_5 + \mathbb{A}_1$	$\frac{3}{5}$
2	3	$\mathbb{D}_6$	$\frac{1}{2}$
2	2	$\mathbb{D}_6 + \mathbb{A}_1$	$\frac{1}{2}$
2	4	$\mathbb{E}_6$	$\frac{3}{7}$
2	1	$\mathbb{E}_7$	$\frac{3}{10}$

δ-invariants of Du Val del Pezzo surfaces (Denisova)

d	Singularities and additional conditions	δ
1	$\mathbb{A}_1$, $2\mathbb{A}_1$, $3\mathbb{A}_1$, $4\mathbb{A}_1$, $5\mathbb{A}_1$, $6\mathbb{A}_1$, $7\mathbb{A}_1$, $8\mathbb{A}_1$ all curves in $\|-K_S\|$ containing singular points are nodal	2
1	$\mathbb{A}_1$, $2\mathbb{A}_1$, $3\mathbb{A}_1$, $4\mathbb{A}_1$, $5\mathbb{A}_1$, $6\mathbb{A}_1$, $7\mathbb{A}_1$, $8\mathbb{A}_1$ some curve in $\|-K_S\|$ containing singular point is cuspidal	$\frac{9}{5}$
1	$\mathbb{A}_2$, $\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_2+2\mathbb{A}_1$, $\mathbb{A}_2+3\mathbb{A}_1$, $\mathbb{A}_2+4\mathbb{A}_1$, $2\mathbb{A}_2$, $2\mathbb{A}_2+\mathbb{A}_1$, $2\mathbb{A}_2+2\mathbb{A}_1$, $3\mathbb{A}_2$, $3\mathbb{A}_2+\mathbb{A}_1$, $4\mathbb{A}_2$ all curves in $\|-K_S\|$ containing $\mathbb{A}_2$ singularities are nodal	$\frac{12}{7}$
1	$\mathbb{A}_2$, $\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_2+2\mathbb{A}_1$, $\mathbb{A}_2+3\mathbb{A}_1$, $\mathbb{A}_2+4\mathbb{A}_1$, $2\mathbb{A}_2$, $2\mathbb{A}_2+\mathbb{A}_1$, $2\mathbb{A}_2+2\mathbb{A}_1$, $3\mathbb{A}_2$, $3\mathbb{A}_2+\mathbb{A}_1$, $4\mathbb{A}_2$ some curve in $\|-K_S\|$ containing $\mathbb{A}_2$ singularity is cuspidal	$\frac{3}{2}$
1	$\mathbb{A}_3$, $\mathbb{A}_3+\mathbb{A}_1$, $\mathbb{A}_3+2\mathbb{A}_1$, $\mathbb{A}_3+3\mathbb{A}_1$, $\mathbb{A}_3+4\mathbb{A}_1$, $\mathbb{A}_3+\mathbb{A}_2$, $\mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_3+\mathbb{A}_2+2\mathbb{A}_1$, $2\mathbb{A}_3$, $2\mathbb{A}_3+\mathbb{A}_1$, $2\mathbb{A}_3+2\mathbb{A}_1$	$\frac{3}{2}$

δ-invariants of Du Val del Pezzo surfaces (Denisova)

d	Singularities and additional conditions	δ
1	$\mathbb{A}_4$, $\mathbb{A}_4+\mathbb{A}_1$, $\mathbb{A}_4+2\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_2$, $\mathbb{A}_4+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_3$, $2\mathbb{A}_4$	$\frac{4}{3}$
1	$\mathbb{A}_5$, $\mathbb{A}_5+\mathbb{A}_1$, $\mathbb{A}_5+2\mathbb{A}_1$, $\mathbb{A}_5+\mathbb{A}_2$, $\mathbb{A}_5+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_5+\mathbb{A}_3$	$\frac{6}{5}$
1	$\mathbb{A}_6$, $\mathbb{A}_6+\mathbb{A}_1$	$\frac{9}{8}$
1	$\mathbb{A}_7$, $\mathbb{A}_7+\mathbb{A}_1$ and the ramification curve of $S\to\mathbb{P}(1,1,2)$ is irreducible	$\frac{18}{17}$
1	$\mathbb{A}_7$, $\mathbb{A}_7+\mathbb{A}_1$ and the ramification curve of $S\to\mathbb{P}(1,1,2)$ is irreducible	1
1	$\mathbb{A}_8$, $\mathbb{D}_4$, $\mathbb{D}_4+\mathbb{A}_1$, $\mathbb{D}_4+2\mathbb{A}_1$, $\mathbb{D}_4+3\mathbb{A}_1$, $\mathbb{D}_4+4\mathbb{A}_1$,$\mathbb{D}_4+\mathbb{A}_2$, $\mathbb{D}_4+\mathbb{A}_3$, $2\mathbb{D}_4$	1
1	$\mathbb{D}_5$, $\mathbb{D}_5+\mathbb{A}_1$, $\mathbb{D}_5+2\mathbb{A}_1$, $\mathbb{D}_5+\mathbb{A}_2$, $\mathbb{D}_5+\mathbb{A}_3$	$\frac{6}{7}$
1	$\mathbb{D}_6$, $\mathbb{D}_6+\mathbb{A}_1$, $\mathbb{D}_6+2\mathbb{A}_1$	$\frac{3}{4}$
1	$\mathbb{D}_7$	$\frac{2}{3}$
1	$\mathbb{D}_8$, $\mathbb{E}_6$, $\mathbb{E}_6+\mathbb{A}_1$, $\mathbb{E}_6+\mathbb{A}_2$	$\frac{3}{5}$
1	$\mathbb{E}_7$, $\mathbb{E}_7+\mathbb{A}_1$	$\frac{3}{7}$
1	$\mathbb{E}_8$	$\frac{3}{11}$

Cylinders in singular del Pezzo surfaces

Let S be a del Pezzo surface with at most quotient singularities.

Definition

A $(-K_S)$-polar cylinder in S is an open subset

$$U = S \setminus \mathrm{Supp}(D) \simeq C \times \mathbb{A}^1,$$

where C is an affine curve, and D is a $\mathbb{Q}$-divisor with $D \sim_{\mathbb{Q}} -K_S$.

Theorem (C., Kishimoto, Park, Prokhorov, Won, Zaidenberg)

Suppose that S has Du Val singularities. Set $d = (-K_S)^2$
Then S does not admit a $(-K_S)$-polar cylinder if and only if

1. $d = 1$ and S has at most $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{D}_4$ singularities,
2. $d = 2$ and S has at most $\mathbb{A}_1$ singularities,
3. $d = 3$ and S is smooth.

In-kyun Kim and Jaehyun Kim have constructed
singular del Pezzo surfaces with quotient singularities without
anticanonical polar cylinders that are not K-semistable.

K-polystable Du Val Del Pezzo surfaces

Classification of K-polystable smoothable del Pezzo surfaces and description of the corresponding components of the K-moduli has been done by Yuji Odaka, Cristiano Spotti and Song Sun in 2016.

Example (Mukai, Mabuchi)

Let S be a Du Val del Pezzo surface of degree 4.
Then S is K-polystable $\iff$ S is a complete intersection

$$\left\{ \sum_{i=0}^{4} a_i x_i^2 = \sum_{i=0}^{4} b_i x_i^2 = 0 \right\} \subset \mathbb{P}^4_{x_0,x_1,x_2,x_3,x_4}$$

for some a_i and b_j such that one of the following holds:

- either S is smooth,
- or S has two $\mathbb{A}_1$ singularities,
- or S has four $\mathbb{A}_1$ singularities.

These surfaces form a component of the K-moduli space.

K-polystable Du Val del Pezzo surfaces of low degree

Let S be a Du Val del Pezzo surface of degree $d = (-K_S)^2 \leqslant 3$.
If $d = 3$, then S is K-polystable if and only if

- either S has at most $\mathbb{A}_1$ singularities,
- or $S \simeq \{xyz = t^3\} \subset \mathbb{P}^3_{x,y,z,t}$.

Similarly, if $d = 2$, then S is K-polystable if and only if

- either S has at most $\mathbb{A}_1$ or $\mathbb{A}_2$ singularities,
- or $S \simeq \{w^2 = (xy - z^2)(xy - \lambda z^2)\} \subset \mathbb{P}(1_x, 1_y, 1_z, 2_w)$ for some $\lambda \neq 1$.

Finally, if $d = 1$, then S is K-polystable if and only if

- either S has at most $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_5$, $\mathbb{A}_6$ singularities,
- or S has singular point of type $\mathbb{A}_7$, and
  $$S \not\simeq \{w^2 = z\big(z^2 + z(x^2 + \lambda y^2) + y^4\big)\} \subset \mathbb{P}(1_x, 1_y, 2_z, 3_w)$$
  for any $\lambda \neq \pm 2$,
- or $S \simeq \{w^2 = z(z^2 + xy)(z^2 + \lambda xy)\}$ for some $\lambda \notin \{0, 1\}$.

Del Pezzo surfaces of Gorenstein index two and three

Let S be a del Pezzo surface with at most quotient singularities.

- Let I be the smallest natural such that $I(-K_S)$ is Cartier.
- Then $I = 1 \iff S$ has at most Du Val singularities.

If $I = 2$, all possibilities for S have been described in

Del Pezzo and K3 surfaces

by Valera Alexeev and Slava Nikulin (1988,2006), and in

Classification of log del Pezzo surfaces of index two

by Noboru Nakayama (2007) without using K3 surfaces.
If $I = 3$, all possibilities for S have been described in

Classification of log del Pezzo surfaces of index three

by Kento Fujita and Kazunori Yasutake (2017).

- K-stability of these surfaces has not been explored yet.

Del Pezzo surfaces with $\mathbb{C}^*$-action

Let S be a del Pezzo surface with at most quotient singularities.

- Suppose that $\mathrm{Aut}(S)$ contains $\mathbb{C}^*$.

If $\mathrm{Pic}(S) \simeq \mathbb{Z}$, the algorithm to find S with fixed I is given in

Del Pezzo surfaces of Picard number one admitting a torus action

by Daniel Haettig, Beatrice Hafner, Jürgen Hausen, Justus Springer.
K-stability of these surfaces has been studied in

Log del Pezzo $\mathbb{C}^*$-surfaces, Kähler–Einstein metrics, Kähler–Ricci solitons and Sasaki–Einstein metrics

by Daniel Hättig, Jürgen Hausen and Hendrik Süß (2023).

Example

If S is toric and $I \leqslant 10$, the number of possibilities for S is given in

I	1	2	3	4	5	6	7	8	9	10
All	16	30	99	91	250	379	429	307	690	916
K-ps	5	1	4	2	7	4	10	3	7	4

Johnson–Kollár surfaces

Let S be a quasismooth well-formed surface in $\mathbb{P}(a_1, a_2, a_3, a_4)$.
Set $I = a_1 + a_2 + a_3 + a_4 - d$, where d is the degree of S.
Suppose that $I \geqslant 1$ and $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$.

Theorem (Johnson, Kollár)

If $I = 1$, then either

$$\big(a_1, a_2, a_3, a_4, d\big) = (2, 2n+1, 2n+1, 4n+1, 8n+4)$$

for some positive integer n, or (a_1, a_2, a_3, a_4, d) is one of

$$(1,1,1,1,3),(1,1,1,2,4),(1,1,2,3,6),(1,2,3,5,10),(1,3,5,7,15),(1,3,5,8,16),$$
$$(2,3,5,9,18),(3,3,5,5,15),(3,5,7,11,25),(3,5,7,14,28),(3,5,11,18,36),$$
$$(5,14,17,21,56),(5,19,27,31,81),(5,19,27,50,100),(7,11,27,37,81),$$
$$(7,11,27,44,88),(9,15,17,20,60),(9,15,23,23,69),(11,29,39,49,127),$$
$$(11,49,69,128,256),(13,23,35,57,127),(13,35,81,128,256).$$

If $I = 1$, S is K-stable (Araujo, C., Johnson, Kollár, Park, Shramov).
The corresponding K-moduli components have not been studied yet.

Weighted del Pezzo hypersurfaces of higher index

Following Johnson and Kollár, Erik Paemurru found an algorithm that gives all possibilities for (a_1, a_2, a_3, a_4, d) for fixed $I \geqslant 1$.

If $I = 2$, then $(a_1, a_2, a_3, a_4, d) = (1, 1, s, r, s + r)$ for some $s, r > 0$, or (a_1, a_2, a_3, a_4, d) is one of the quintuples

$$(1, 2, m, m + 1, 2m + 2), (1, 3, 3m, 3m + 1, 6m + 3),$$
$$(1, 3, 3m + 1, 3m + 2, 6m + 5), (3, 3m, 3m + 1, 3m + 1, 9m + 3),$$
$$(3, 3m + 1, 3m + 2, 3m + 2, 9m + 6), (3, 3m + 1, 6m + 1, 9m, 18m + 3),$$
$$(3, 3m + 1, 6m + 1, 9m + 3, 18m + 6), (4, 2m + 1, 4m + 2, 6m + 1, 12m + 6),$$
$$(4, 2m + 3, 2m + 3, 4m + 4, 8m + 12), (3, 3m + 4, 3m + 5, 6m + 7, 12m + 17)$$

for some integer $m \geqslant 1$, or (a_1, a_2, a_3, a_4, d) is one of

$$(1, 4, 5, 7, 15), (1, 4, 5, 8, 16), (1, 5, 7, 11, 22), (1, 6, 9, 13, 27), (1, 7, 12, 18, 36),$$
$$(1, 8, 13, 20, 40), (1, 9, 15, 22, 45), (1, 3, 4, 6, 12), (1, 4, 6, 9, 18), (1, 6, 10, 15, 30),$$
$$(2, 3, 4, 7, 14), (3, 4, 5, 10, 20), (3, 4, 10, 15, 30), (3, 4, 6, 7, 18), (5, 13, 19, 22, 57),$$
$$(5, 13, 19, 35, 70), (6, 9, 10, 13, 36), (7, 8, 19, 25, 57), (7, 8, 19, 32, 64), (9, 12, 13, 16, 48),$$
$$(9, 12, 19, 19, 57), (9, 19, 24, 31, 81), (10, 19, 35, 43, 105), (11, 21, 28, 47, 105),$$
$$(11, 25, 32, 41, 107), (11, 25, 34, 43, 111), (11, 43, 61, 113, 226), (13, 18, 45, 61, 135),$$
$$(13, 20, 29, 47, 107), (13, 20, 31, 49, 111), (13, 31, 71, 113, 226), (14, 17, 29, 41, 99).$$

K-polystable weighted del Pezzo hypersurfaces

Let S be a quasismooth well-formed surface in $\mathbb{P}(a_1, a_2, a_3, a_4)$.
Set $I = a_1 + a_2 + a_3 + a_4 - d$, where d is the degree of S.
Suppose that $I \geqslant 1$ and $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$.

Theorem (Kim, Won, Viswanathan)

Suppose that $I = 2$. Then S is not K-polystable if and only if

- either $(a_1, a_2, a_3, a_4, d) = (1, 1, s, r, s + r)$ for $s \neq r > 0$,
- or $(a_1, a_2, a_3, a_4, d) = (1, 3, 3n + 3, 3n + 4, 6n + 9)$ for $n > 0$,
- or $(a_1, a_2, a_3, a_4, d) = (1, 3, 3n + 4, 3n + 5, 6n + 11)$ for$n > 0$,
- or $(a_1, a_2, a_3, a_4, d) \in \{(1, 6, 9, 13, 27), (1, 9, 15, 22, 45)\}$.

Theorem (Kim, Won, Viswanathan)

Suppose that $I = 3$. Then S is K-polystable if and only if $a_1 > 2$.

Conjecture (Kim, Won, Viswanathan)

Suppose that $I \gg 0$. Then S is K-polystable if and only if $a_1 > \frac{2}{3}I$.

Weighted del Pezzo complete intersections

Let S be a quasismooth well-formed complete intersection

$$S = X_{d_1} \cap X_{d_2} \subset \mathbb{P}(a_1, a_2, a_3, a_4, a_5),$$

where X_{d_1} and X_{d_2} are hypersurfaces of degrees d_1 and d_2.
Set $I = a_1 + a_2 + a_3 + a_4 + a_5 - d_1 - d_2$. Suppose that $I \geqslant 1$.

Remark

All possible values of $(a_1, a_2, a_3, a_4, a_5, d_1, d_2)$ have been found in

Weighted complete intersection del Pezzo surfaces

by Evgeny Mayanskiy (unpublished).

Theorem (Kim, Park)

If $I = 1$, then S is K-polystable.

- For $I \geqslant 2$, K-stability of S has been unexplored.

$\mathbb{Q}$-homology planes

- Let S be a del Pezzo surface with KLT singularities.
- Suppose that $\mathrm{rk}\,\mathrm{Pic}(S) = 1$.

Up to a bounded family, all possibilities for S have been found in

Rational curves on quasi-projective surfaces

by Sean Keel and James McKernan (1999), who wrote

> *We believe that repeated application of the same methods would eventually yield a complete classification.*

We know that $|\mathrm{Sing}(S)| \leqslant 4$ (Belousov, 2009).
If $|\mathrm{Sing}(S)| = 1$, then all possibilities for S have been found in

Normal log canonical del Pezzo surfaces of rank one with unique singular points

by Hideo Kojima (2014).
Recently, Justin Lacini found all possibilities for S in

On rank one log del Pezzo surfaces in characteristic different from two and three.

K-stability of these surfaces has not been explored yet.

Smooth Fano 3-folds

Smooth Fano 3-folds has been classified in

105 deformation families

by Iskovskikh, Mori and Mukai.
For 87 families, K-polystable smooth members have been found by

Abban, Araujo, Arezzo, Belousov, Castravet, C., Denisova, Dervan, Donaldson, Fujita, Ghigi, Giovenzana, Guerreiro, Ilten, Kaloghiros, Kishimoto, Liu, Loginov, Malbon, Martinez-Garcia, Nadel, Park, Pirola, Shramov, Spotti, Suess, Sun, Tian, Viswanathan, Xu, Zhao, Zhuang.

The remaining 18 families are

№1.9, №1.10, №2.5, №2.9, №2.10, №2.11, №2.12, №2.13, №2.14, №2.16, №2.17, №2.20, №3.2, №3.5, №3.6, №3.7, №3.8, №3.11.

We know that the 27 deformation families

№2.23, №2.26, №2.28, №2.30, №2.31, №2.33, №2.35, №2.36, №3.14, №3.16, №3.18, №3.21, №3.22, №3.23, №3.24, №3.26, №3.28, №3.29, №3.30, №3.31, №4.5, №4.8, №4.9, №4.10, №4.11, №4.12, №5.2

do not contains K-polystable smooth members.
General members of the remaining 78 families are K-polystable.

Iskovskikh–Mori–Mukai's classification

List of Fano varieties—Fanograph

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Fano threefolds with $\rho = 1$

ID	$-K_X^3$	g	$h^{1,2}$	index	description	blowups	rational	unirational	moduli	Aut^0
1-1	2	2	52	1	double cover of $\mathbb{P}^3$ with branch locus a divisor of degree 6 **alternative** hypersurface of degree 6 in $\mathbb{P}(1,1,1,1,3)$		no	?	68	0
1-2	4	3	30	1	a) hypersurface of degree 4 in $\mathbb{P}^4$ b) double cover of 1-16 with branch locus a divisor of degree 8		no	some	a) 45 b) 44	0
1-3	6	4	20	1	complete intersection of quadric and cubic in $\mathbb{P}^5$		no	yes	34	0
1-4	8	5	14	1	complete intersection of 3 quadrics in $\mathbb{P}^6$		no	yes	27	0
1-5	10	6	10	1	**Gushel–Mukai 3-fold** a) section of Plücker embedding of $\mathrm{Gr}(2,5)$ by codimension 2 subspace and a quadric b) double cover of 1-15 with branch locus an anticanonical divisor		generically non-rational	yes	a) 22 b) 19	0
1-6	12	7	7	1	section of half-spinor embedding of a connected component of $\mathrm{OGr}_+(5,10)$ by codimension 7 subspace		yes	yes	18	0
1-7	14	8	5	1	section of Plücker embedding of $\mathrm{Gr}(2,6)$ by codimension 5 subspace		no	yes	15	0
1-8	16	9	3	1	section of Plücker embedding of $\mathrm{SGr}(3,6)$ by codimension 3 subspace		yes	yes	12	0
1-9	18	10	2	1	section of the adjoint G_2-Grassmannian $G_2\mathrm{Gr}(2,7)$ by codimension 2 subspace		yes	yes	10	0

$\mathrm{Aut}^0(X)$	moduli
PGL_2	0

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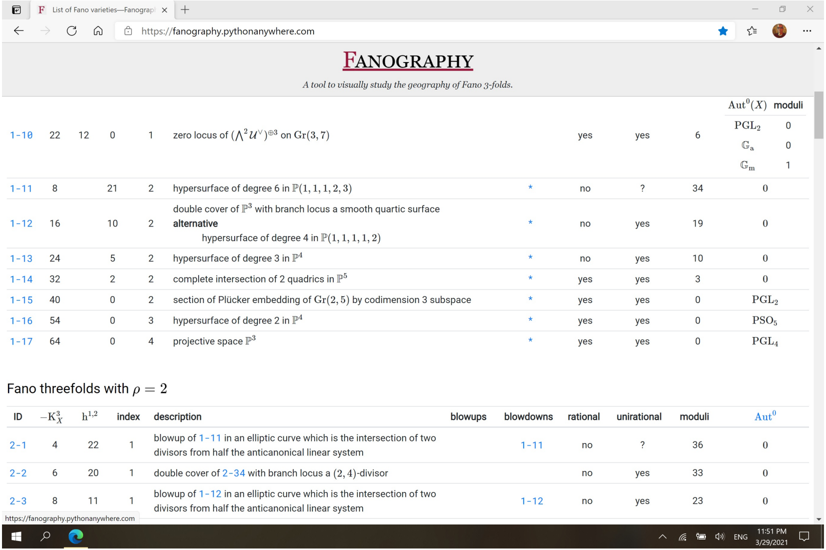

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1-10	22	12	0	1	zero locus of $(\bigwedge^2 \mathcal{U}^\vee)^{\oplus 3}$ on $\mathrm{Gr}(3,7)$		yes	yes	6	$\mathrm{Aut}^0(X)$ moduli PGL_2 0 $\mathbb{G}_a$ 0 $\mathbb{G}_m$ 1
1-11	8		21	2	hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$	*	no	?	34	0
1-12	16		10	2	double cover of $\mathbb{P}^3$ with branch locus a smooth quartic surface **alternative** hypersurface of degree 4 in $\mathbb{P}(1,1,1,1,2)$	*	no	yes	19	0
1-13	24		5	2	hypersurface of degree 3 in $\mathbb{P}^4$	*	no	yes	10	0
1-14	32		2	2	complete intersection of 2 quadrics in $\mathbb{P}^5$	*	yes	yes	3	0
1-15	40		0	2	section of Plücker embedding of $\mathrm{Gr}(2,5)$ by codimension 3 subspace	*	yes	yes	0	PGL_2
1-16	54		0	3	hypersurface of degree 2 in $\mathbb{P}^4$	*	yes	yes	0	PSO_5
1-17	64		0	4	projective space $\mathbb{P}^3$	*	yes	yes	0	PGL_4

Fano threefolds with $\rho = 2$

ID	$-K_X^3$	$h^{1,2}$	index	description	blowups	blowdowns	rational	unirational	moduli	Aut^0
2-1	4	22	1	blowup of 1-11 in an elliptic curve which is the intersection of two divisors from half the anticanonical linear system		1-11	no	?	36	0
2-2	6	20	1	double cover of 2-34 with branch locus a $(2,4)$-divisor			no	yes	33	0
2-3	8	11	1	blowup of 1-12 in an elliptic curve which is the intersection of two divisors from half the anticanonical linear system		1-12	no	yes	23	0

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2-4	10	10	1	blowup of 1-17 in the intersection of two cubics **alternative** $(1,3)$-divisor on $\mathbb{P}^1 \times \mathbb{P}^3$	1-17	yes	yes	21	0
2-5	12	6	1	blowup of 1-13 in a plane cubic	1-13	no	yes	16	0
2-6	12	9	1	**Verra 3-fold** a) $(2,2)$-divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ b) double cover of 2-32 with branch locus an anticanonical divisor		no	yes	a) 19 b) 18	0
2-7	14	5	1	blowup of 1-16 in the intersection of two divisors from $\lvert\mathcal{O}_Q(2)\rvert$	1-16	yes	yes	14	0
2-8	14	9	1	a) double cover of 2-35 with branch locus an anticanonical divisor such that the intersection with the exceptional divisor is smooth b) double cover of 2-35 with branch locus an anticanonical divisor such that the intersection with the exceptional divisor is singular but reduced		no	yes	a) 18 b) 17	0
2-9	16	5	1	complete intersection of degree $(1,1)$ and $(2,1)$ in $\mathbb{P}^3 \times \mathbb{P}^2$ **alternative** blowup of 1-17 in a curve of degree 7 and genus 5, which is an intersection of 3 cubics	1-17	yes	yes	13	0
2-10	16	3	1	blowup of 1-14 in an elliptic curve which is an intersection of 2 hyperplanes	1-14	yes	yes	11	0
2-11	18	5	1	blowup of 1-13 in a line	1-13	no	yes	12	0
2-12	20	3	1	intersection of 3 $(1,1)$-divisors in $\mathbb{P}^3 \times \mathbb{P}^3$ **alternative** blowup of 1-17 in a curve of degree 6 and genus 3 which is an intersection of 4 cubics	1-17	yes	yes	9	0
2-13	20	2	1	blowup of 1-16 in a curve of degree 6 and genus 2	1-16	yes	yes	8	0

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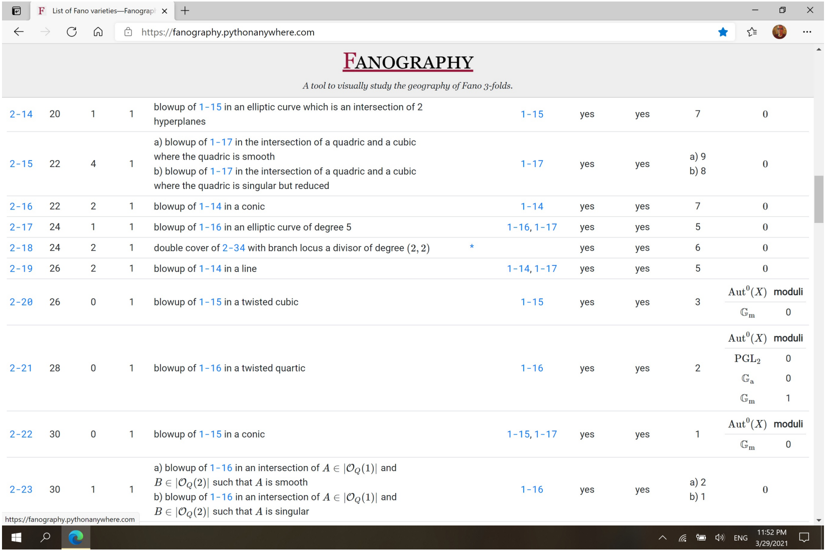

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2-14	20	1	1	blowup of 1-15 in an elliptic curve which is an intersection of 2 hyperplanes		1-15	yes	yes	7	0
2-15	22	4	1	a) blowup of 1-17 in the intersection of a quadric and a cubic where the quadric is smooth b) blowup of 1-17 in the intersection of a quadric and a cubic where the quadric is singular but reduced		1-17	yes	yes	a) 9 b) 8	0
2-16	22	2	1	blowup of 1-14 in a conic		1-14	yes	yes	7	0
2-17	24	1	1	blowup of 1-16 in an elliptic curve of degree 5		1-16, 1-17	yes	yes	5	0
2-18	24	2	1	double cover of 2-34 with branch locus a divisor of degree $(2,2)$	*		yes	yes	6	0
2-19	26	2	1	blowup of 1-14 in a line		1-14, 1-17	yes	yes	5	0
2-20	26	0	1	blowup of 1-15 in a twisted cubic		1-15	yes	yes	3	$\mathrm{Aut}^0(X)$ moduli $\mathbb{G}_m$ 0
2-21	28	0	1	blowup of 1-16 in a twisted quartic		1-16	yes	yes	2	$\mathrm{Aut}^0(X)$ moduli PGL_2 0 $\mathbb{G}_a$ 0 $\mathbb{G}_m$ 1
2-22	30	0	1	blowup of 1-15 in a conic		1-15, 1-17	yes	yes	1	$\mathrm{Aut}^0(X)$ moduli $\mathbb{G}_m$ 0
2-23	30	1	1	a) blowup of 1-16 in an intersection of $A \in \lvert\mathcal{O}_Q(1)\rvert$ and $B \in \lvert\mathcal{O}_Q(2)\rvert$ such that A is smooth b) blowup of 1-16 in an intersection of $A \in \lvert\mathcal{O}_Q(1)\rvert$ and $B \in \lvert\mathcal{O}_Q(2)\rvert$ such that A is singular		1-16	yes	yes	a) 2 b) 1	0

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										$\mathrm{Aut}^0(X)$	moduli
2-24	30	0	1	divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,2)$	*		yes	yes	1	$\mathbb{G}_m^2$ $\mathbb{G}_m$	0 0
2-25	32	1	1	blowup of 1-17 in an elliptic curve which is an intersection of 2 quadrics **alternative** $(1,2)$-divisor on $\mathbb{P}^1 \times \mathbb{P}^3$	*	1-17	yes	yes	1	0	
										$\mathrm{Aut}^0(X)$	moduli
2-26	34	0	1	blowup of 1-15 in a line		1-15, 1-16	yes	yes	0	B $\mathbb{G}_m$	0 0
2-27	38	0	1	blowup of 1-17 in a twisted cubic	*	1-17	yes	yes	0	PGL_2	
2-28	40	1	1	blowup of 1-17 in a plane cubic		1-17	yes	yes	1	$\mathbb{G}_a^3 \rtimes \mathbb{G}_m$	
2-29	40	0	1	blowup of 1-16 in a conic	*	1-16	yes	yes	0	$\mathbb{G}_m \times \mathrm{PGL}_2$	
2-30	46	0	1	blowup of 1-17 in a conic	*	1-17	yes	yes	0	$\mathrm{PSO}_{5;1}$	
2-31	46	0	1	blowup of 1-16 in a line	*	1-16	yes	yes	0	$\mathrm{PSO}_{5;2}$	
2-32	48	0	2	divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$ **alternative** $\mathbb{P}(\mathrm{T}_{\mathbb{P}^2})$ the complete flag variety for $\mathbb{P}^2$	*		yes	yes	0	PGL_3	
2-33	54	0	1	blowup of 1-17 in a line	*	1-17	yes	yes	0	$\mathrm{PGL}_{4;2}$	
2-34	54	0	1	$\mathbb{P}^1 \times \mathbb{P}^2$	*		yes	yes	0	$\mathrm{PGL}_2 \times \mathrm{PGL}_3$	
				$\mathrm{Bl}_p\mathbb{P}^3$ **alternative**	*		yes	yes	0	$\mathrm{PGL}_{4;1}$	

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ID	$-K_X^3$	$h^{1,2}$	index	description	blowups	blowdowns	rational	unirational	moduli	Aut^0
2-35	56	0	2	$\mathrm{Bl}_p\mathbb{P}^3$ **alternative** $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$	*		yes	yes	0	$\mathrm{PGL}_{4;1}$
2-36	62	0	1	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$	*		yes	yes	0	$\mathrm{Aut}(\mathbb{P}(1,1,1,2))$

Fano threefolds with $\rho = 3$

ID	$-K_X^3$	$h^{1,2}$	index	description	blowups	blowdowns	rational	unirational	moduli	Aut^0
3-1	12	8	1	double cover of 3-27 with branch locus a divisor of degree $(2,2,2)$			no	yes	17	0
3-2	14	3	1	divisor from $\lvert\mathcal{L}^{\otimes 2} \otimes \mathcal{O}(2,3)\rvert$ on the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1,-1)^{\oplus 2})$ over $\mathbb{P}^1 \times \mathbb{P}^1$ such that $X \cap Y$ is irreducible, and $\mathcal{L}$ is the tautological bundle, and $Y \in \lvert\mathcal{L}\rvert$			yes	yes	11	0
3-3	18	3	1	divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of degree $(1,1,2)$		2-34	yes	yes	9	0
3-4	18	2	1	blowup of 2-18 in a smooth fiber of the composition of the projection to $\mathbb{P}^1 \times \mathbb{P}^2$ with the projection to $\mathbb{P}^2$ of the double cover with the projection		2-18	yes	yes	8	0
3-5	20	0	1	blowup of 2-34 in a curve C of degree $(5,2)$ such that $C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ is an embedding		2-34	yes	yes	5	$\mathrm{Aut}^0(X)$ / moduli: $\mathbb{G}_m$ / 0
3-6	22	1	1	blowup of 1-17 in the disjoint union of a line and an elliptic curve of degree 4 **alternative** complete intersection of degree $(1,0,2)$ and $(0,1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$		2-25, 2-33	yes	yes	5	0

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3-7	24	1	1	blowup of 2-32 in an elliptic curve which is the intersection of two divisors from $\lvert -\frac{1}{2}\mathrm{K}_W\rvert$	2-32, 2-34	yes	yes	4	0
3-8	24	0	1	divisor from the linear system $\lvert(\alpha\circ\pi_1)^*(\mathcal{O}_{\mathbb{P}^2}(1)\otimes\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(2))\rvert$ where $\pi_i\colon \mathrm{Bl}_1\times\mathbb{P}^2$ are the projections, and $\alpha\colon \mathrm{Bl}_1\mathbb{P}^2\to\mathbb{P}^2$ is the blowup	2-24, 2-34	yes	yes	3	$\mathrm{Aut}^0(X)$ moduli $\mathbb{G}_\mathrm{m}$ 0
3-9	26	3	1	blowup of the cone over the Veronese of $\mathbb{P}^2$ in $\mathbb{P}^5$ with center the disjoint union of the vertex and a quartic curve on $\mathbb{P}^2$	2-36	yes	yes	6	$\mathbb{G}_\mathrm{m}$
3-10	26	0	1	blowup of 1-16 in the disjoint union of 2 conics **alternative** complete intersection of degree $(1,0,1)$, $(0,1,1)$ and $(0,0,2)$ in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^4$	2-29	yes	yes	2	$\mathrm{Aut}^0(X)$ moduli $\mathbb{G}_\mathrm{m}^2$ 0 $\mathbb{G}_\mathrm{m}$ 1
3-11	28	1	1	blowup of 2-35 in an elliptic curve which is the intersection of two divisors from $\lvert -\frac{1}{2}\mathrm{K}_{V_7}\rvert$	2-25, 2-34, 2-35	yes	yes	2	0
3-12	28	0	1	blowup of 1-17 in the disjoint union of a line and a twisted cubic	2-27, 2-33, 2-34	yes	yes	1	$\mathrm{Aut}^0(X)$ moduli $\mathbb{G}_\mathrm{m}$ 0
3-13	30	0	1	blowup of 2-32 in a curve C of bidegree $(2,2)$ such that the composition $C\hookrightarrow W\hookrightarrow\mathbb{P}^2\times\mathbb{P}^2\xrightarrow{p_i}\mathbb{P}^2$ is an embedding for $i=1,2$	2-32	yes	yes	1	$\mathrm{Aut}^0(X)$ moduli PGL_2 0 $\mathbb{G}_\mathrm{a}$ 0 $\mathbb{G}_\mathrm{m}$ 1
3-14	32	1	1	blowup of 1-17 in the disjoint union of a plane cubic curve and a point outside the plane	2-35, 2-36	yes	yes	1	$\mathbb{G}_\mathrm{m}$
3-15	32	0	1	blowup of 1-16 in the disjoint union of a line and a conic	2-29, 2-31, 2-34	yes	yes	0	$\mathbb{G}_\mathrm{m}$

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3-16	34	0	1	blowup of 2-35 in the proper transform of a twisted cubic containing the center of the blowup		2-27, 2-32, 2-35	yes	yes	0	B
3-17	36	0	1	divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of degree $(1,1,1)$	*	2-34	yes	yes	0	PGL_2
3-18	36	0	1	blowup of 1-17 in the disjoint union of a line and a conic	*	2-29, 2-30, 2-33	yes	yes	0	$\mathrm{B} \times \mathbb{G}_\mathrm{m}$
3-19	38	0	1	blowup of 1-16 in two non-collinear points	*	2-35	yes	yes	0	$\mathbb{G}_\mathrm{m} \times \mathrm{PGL}_2$
3-20	38	0	1	blowup of 1-16 in the disjoint union of two lines		2-31, 2-32	yes	yes	0	$\mathbb{G}_\mathrm{m} \times \mathrm{PGL}_2$
3-21	38	0	1	blowup of 2-34 in a curve of degree $(2,1)$	*	2-34	yes	yes	0	$\mathbb{G}_\mathrm{a}^2 \rtimes \mathbb{G}_\mathrm{m}^2$
3-22	40	0	1	blowup of 2-34 in a conic on $\{x\} \times \mathbb{P}^2, x \in \mathbb{P}^1$		2-34, 2-36	yes	yes	0	$\mathrm{B} \times \mathrm{PGL}_2$
3-23	42	0	1	blowup of 2-35 in the proper transform of a conic containing the center of the blowup **alternative** complete intersection of degree $(1,1,0)$ and $(0,1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$		2-30, 2-31, 2-35	yes	yes	0	$\mathbb{G}_\mathrm{a}^3 \rtimes (\mathrm{B} \times \mathbb{G}_\mathrm{m})$
3-24	42	0	1	the fiber product of 2-32 with $\mathrm{Bl}_p\mathbb{P}^2$ over $\mathbb{P}^2$	*	2-32, 2-34	yes	yes	0	$\mathrm{PGL}_{3;1}$
3-25	44	0	1	blowup of 1-17 in the disjoint union of two lines **alternative** $\mathbb{P}(\mathcal{O}(1,0) \oplus \mathcal{O}(0,1))$ over $\mathbb{P}^1 \times \mathbb{P}^1$	*	2-33	yes	yes	0	$\mathrm{PGL}_{(2,2)}$
3-26	46	0	1	blowup of 1-17 in the disjoint union of a point and a line **alternative** blowup of line on a plane which is section of 2-34 mapping to $\mathbb{P}^2$	*	2-34, 2-35	yes	yes	0	$\mathbb{G}_\mathrm{a}^3 \rtimes (\mathrm{GL}_2 \times \mathbb{G}_\mathrm{m})$
3-27	48	0	2	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	*		yes	yes	0	PGL_2^3
				$\mathbb{P}^1 \times \mathrm{Bl}_p\mathbb{P}^2$	*	2-34	yes	yes	0	$\mathrm{PGL}_2 \times \mathrm{PGL}_{3;1}$

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3-29	50	0	1	blowup of 2-35 in a line on the exceptional divisor		2-35	yes	yes	0	$\mathrm{PGL}_{4;3,1}$
3-30	50	0	1	blowup of 2-35 in the proper transform of a line containing the center of the blowup **alternative** $\mathbb{P}_{\mathbb{F}_1}(\mathcal{O}\oplus\mathcal{O}(\ell))$ where $\ell^2=1$	*	2-33, 2-35	yes	yes	0	$\mathrm{PGL}_{4;2,1}$
3-31	52	0	1	blowup of the cone over a smooth quadric in $\mathbb{P}^3$ in the vertex **alternative** $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1,1))$ over $\mathbb{P}^1\times\mathbb{P}^1$	*		yes	yes	0	$\mathrm{PSO}_{6;1}$

Fano threefolds with $\rho = 4$

ID	$-K_X^3$	$h^{1,2}$	description	blowups	blowdowns	rational	unirational	moduli	Aut^0
4-1	24	1	divisor on $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ of degree $(1,1,1,1)$		3-27	yes	yes	3	0
4-2	28	1	blowup of the cone over a smooth quadric in $\mathbb{P}^3$ in the disjoint union of the vertex and an elliptic curve on the quadric		3-31	yes	yes	2	$\mathbb{G}_m$
4-3	30	0	blowup of 3-27 in a curve of degree $(1,1,2)$		3-17, 3-27, 3-28	yes	yes	0	$\mathbb{G}_m$
4-4	32	0	blowup of 3-19 in the proper transform of a conic through the points	*	3-18, 3-19, 3-30	yes	yes	0	$\mathbb{G}_m^2$
4-5	32	0	blowup of 2-34 in the disjoint union of a curve of degree $(2,1)$ and a curve of degree $(1,0)$		3-21, 3-28, 3-31	yes	yes	0	$\mathbb{G}_m^2$
4-6	34	0	blowup of 1-17 in the disjoint union of 3 lines **alternative** blowup of 3-27 in the tridiagonal		3-25, 3-27	yes	yes	0	PGL_2

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4-6	34	0	blowup of 1-17 in the disjoint union of 3 lines **alternative** blowup of 3-27 in the tridiagonal		3-25, 3-27	yes	yes	0	PGL_2
4-7	36	0	blowup of 2-32 in the disjoint union of a curve of degree $(0,1)$ and a curve of degree $(1,0)$		3-24, 3-28	yes	yes	0	GL_2
4-8	38	0	blowup of 3-27 in a curve of degree $(0,1,1)$		3-27, 3-31	yes	yes	0	$\mathrm{B} \times \mathrm{PGL}_2$
4-9	40	0	blowup of 3-25 in an exceptional curve of the blowup	*	3-25, 3-26, 3-28, 3-30	yes	yes	0	$\mathrm{PGL}_{(2,2);1}$
4-10	42	0	$\mathbb{P}^1 \times \mathrm{Bl}_2\mathbb{P}^2$	*	3-27, 3-28	yes	yes	0	$\mathrm{PGL}_2 \times \mathrm{B}^2$
4-11	44	0	blowup of 3-28 in $\{x\} \times E$, $x \in \mathbb{P}^1$ and E the (-1)-curve	*	3-28, 3-31	yes	yes	0	$\mathrm{B} \times \mathrm{PGL}_{3;1}$
4-12	46	0	blowup of 2-33 in the disjoint union of two exceptional lines of the blowup	*	3-30	yes	yes	0	$\mathbb{G}_a^4 \rtimes (\mathrm{GL}_2 \times \mathbb{G}_m)$
4-13	26	0	blowup of 3-27 in a curve of degree $(1,1,3)$		3-27, 3-31	yes	yes	1	$\mathrm{Aut}^0(X)$ moduli: $\mathbb{G}_m$ 0

Fano threefolds with $\rho = 5$

ID	$-K_X^3$	$h^{1,2}$	description	blowups	blowdowns	rational	unirational	moduli	Aut^0
5-1	28	0	blowup of 2-29 in the disjoint union of three exceptional lines of the blowup		4-4, 4-12	yes	yes	0	$\mathbb{G}_m$
5-2	36	0	blowup of 3-25 in the disjoint union of two exceptional lines on the same irreducible component		4-9, 4-11, 4-12	yes	yes	0	$\mathbb{G}_m \times \mathrm{GL}_2$
5-3	36	0	$\mathbb{P}^1 \times \mathrm{Bl}_3\mathbb{P}^2$	*	4-10	yes	yes	0	$\mathrm{PGL}_2 \times \mathbb{G}_m^2$

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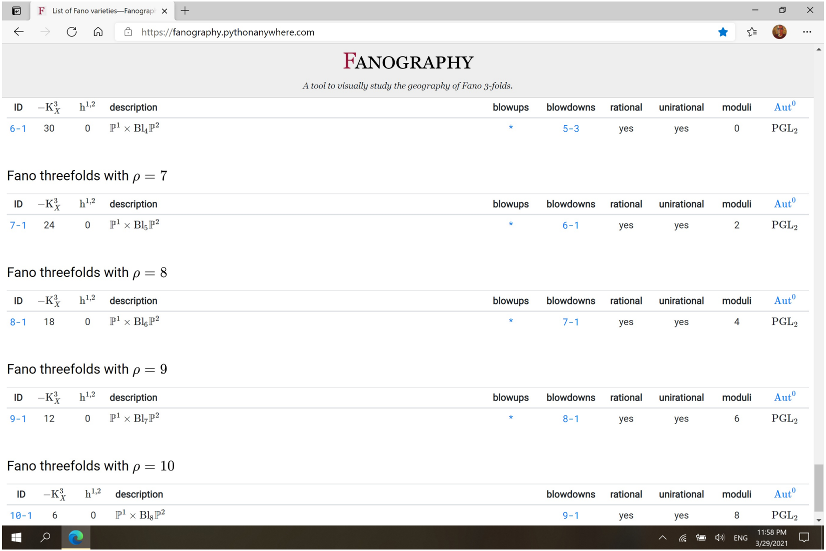

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ID	$-K_X^3$	$h^{1,2}$	description	blowups	blowdowns	rational	unirational	moduli	Aut^0
6-1	30	0	$\mathbb{P}^1 \times \mathrm{Bl}_4\mathbb{P}^2$	*	5-3	yes	yes	0	PGL_2

Fano threefolds with $\rho = 7$

ID	$-K_X^3$	$h^{1,2}$	description	blowups	blowdowns	rational	unirational	moduli	Aut^0
7-1	24	0	$\mathbb{P}^1 \times \mathrm{Bl}_5\mathbb{P}^2$	*	6-1	yes	yes	2	PGL_2

Fano threefolds with $\rho = 8$

ID	$-K_X^3$	$h^{1,2}$	description	blowups	blowdowns	rational	unirational	moduli	Aut^0
8-1	18	0	$\mathbb{P}^1 \times \mathrm{Bl}_6\mathbb{P}^2$	*	7-1	yes	yes	4	PGL_2

Fano threefolds with $\rho = 9$

ID	$-K_X^3$	$h^{1,2}$	description	blowups	blowdowns	rational	unirational	moduli	Aut^0
9-1	12	0	$\mathbb{P}^1 \times \mathrm{Bl}_7\mathbb{P}^2$	*	8-1	yes	yes	6	PGL_2

Fano threefolds with $\rho = 10$

ID	$-K_X^3$	$h^{1,2}$	description	blowdowns	rational	unirational	moduli	Aut^0
10-1	6	0	$\mathbb{P}^1 \times \mathrm{Bl}_8\mathbb{P}^2$	9-1	yes	yes	8	PGL_2

Example: smooth del Pezzo 3-folds

Let X be a smooth del Pezzo 3-fold of degree $d = \frac{1}{8}(-K_X)^3$.
Then $1 \leqslant d \leqslant 8$, and one of the following 9 cases holds:

- $d = 1$ and X is a sextic hypersurface in $\mathbb{P}(1, 1, 1, 2, 3)$;
- $d = 2$ and X is a quartic hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$;
- $d = 3$ and X is a cubic hypersurface in $\mathbb{P}^4$;
- $d = 4$ and X is a complete intersection of two quadrics in $\mathbb{P}^5$;
- $d = 5$ and X is a section of the Grassmannian

  $$\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$$

  in its Plücker embedding by a linear space of dimension 6;
- $d = 6$ and $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$;
- $d = 6$ and X is a divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of degree $(1, 1)$;
- $d = 7$ and X is a blow up of $\mathbb{P}^3$ in one point;
- $d = 8$ and $X = \mathbb{P}^3$.

If $d \neq 7$, then X is K-polystable (Abban, Arezzo, C., Dervan, Ghigi, Liu, Nadel, Pirola, Shramov, Xu, Zhuang).

Example: K-polystable limits of del Pezzo 3-folds

Let X be a K-polystable limit of smooth del Pezzo 3-folds.
Set $d = \frac{1}{8}(-K_X)^3$.

- If $d = 1$, we do not know exactly what X could be.
- If $d = 2$, Ascher, DeVleming and Liu showed that either

$$X = \{w^2 = f_4(x, y, z, t)\} \subset \mathbb{P}(1_x, 1_y, 1_z, 1_w, 2_w),$$

  where f_4 is a GIT-polystable quartic form that is not contained in the orbit of the polynomial $(xz + y^2 + t^2)^2 + at^4$ for any parameter $a \in \mathbb{C}$ for the natural $\mathrm{PGL}_4(\mathbb{C})$-action, or X is a double cover of the cone over $\mathbb{P}^1 \times \mathbb{P}^1$, or

$$X = \{z^2 = tw\} \subset \mathbb{P}(1_x, 1_y, 2_z, 4_t, 4_w).$$

- If $d = 3$, Liu and Xu proved that X is a GIT-polystable cubic hypersurface in $\mathbb{P}^4$ for the $\mathrm{PGL}_5(\mathbb{C})$-action.
- If $d = 4$, Spotti and Sun proved that X is a GIT-polystable complete intersection in $\mathbb{P}^5$ of two quadric hypersurfaces for the natural $\mathrm{PGL}_6(\mathbb{C})$-action.

The remaining 18 deformation families

№	Short Description	$(-K_X)^3$
1.9	see `https://www.fanography.info/`	18
1.10	see `https://www.fanography.info/`	22
2.5	blow up of cubic threefold in elliptic curve	12
2.9	blow up of $\mathbb{P}^3$ along curve of degree 7 and genus 5	16
2.10	blow up of V_4 in elliptic curve	16
2.11	blow up of cubic threefold along line	18
2.12	blow up of $\mathbb{P}^3$ along curve of degree 6 and genus 3	20
2.13	blow up of quadric threefold along curve of degree 6 and genus 2	20
2.14	blow up of V_5 in elliptic curve	20
2.16	blow up of V_4 along conic	22
2.17	blow up of quadric threefold along elliptic curve of degree 5	24
2.20	blow up of V_5 along twisted cubic	26
3.2	see `https://www.fanography.info/`	14
3.5	blow up of $\mathbb{P}^1 \times \mathbb{P}^2$ along curve C of degree $(5,2)$	20
3.6	blow up of $\mathbb{P}^3$ along line and elliptic curve of degree 4	22
3.7	blow up of a divisor of degree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ in elliptic curve	24
3.8	see `https://www.fanography.info/`	24
3.11	blow up of V_7 in elliptic curve	28

Here, we used the following notations:

- V_4 is a complete intersection of two quadrics in $\mathbb{P}^5$;
- V_5 be an intersection of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ with a linear subspace of dimension 5;
- V_7 is a blow up of $\mathbb{P}^3$ in a point.

Smooth Fano 3-folds in the family №1.9

Let Y be the smooth complete intersection in $\mathbb{P}^5$ given by

$$\begin{cases} x_0x_2 - x_1^2 + x_4(x_1 + x_3) + x_5(x_0 + x_2) + x_4^2 = 0, \\ x_1x_3 - x_2^2 + x_5(x_2 + x_0) + x_4(x_3 + x_1) + x_5^2 = 0. \end{cases}$$

Set $\Lambda = \{x_4 = x_5 = 0\} \subset \mathbb{P}^5$. Then

$$\Lambda \cap Y = C + L,$$

where C is a twisted cubic, and L is its secant line.

- Let $\widetilde{Y} \to Y$ be the blow up of the curve C.
- Let $\widetilde{L}$ be the proper transform on $\widetilde{Y}$ of the line L.

Then there exists a flopping contraction $Y \to X$ such that

- X is a Fano 3-fold with one Gorenstein terminal singular point,
- $(-K_X)^3 = 18$ and $\mathrm{Pic}(X) \simeq \mathbb{Z}[-K_X]$,
- the group $\mathrm{Aut}(X)$ is finite and contains $\big(\mathbb{Z}/2\mathbb{Z}\big)^2$.

X is K-stable, so general 3-fold in the family №1.9 is K-stable.
We expect that all smooth members of this family are K-stable.

Smooth Fano 3-folds in the family №1.10

- Let $f = xy^3 + yz^3 + zx^3$.
- Let $C = \{f = 0\} \subset \mathbb{P}^2$.
- Let $L_1, L_2, L_3, L_4, L_5, L_6$ be six distinct lines in $\mathbb{P}^2$.
- Let ℓ_i be the linear form such that $L_i = \{\ell_i = 0\}$.

Set $\Gamma = \sum_{i=1}^{6} L_i$. We say that Γ is polar to C if

$$F = \ell_1^4 + \ell_2^4 + \ell_3^4 + \ell_4^4 + \ell_5^4 + \ell_6^4.$$

Consider Γ as an element of $\mathrm{Hilb}_6\left(\check{\mathbb{P}}^2\right)$. Set

$$X = \overline{\left\{\Gamma \in \mathrm{Hilb}_6\left(\check{\mathbb{P}}^2\right) \;\middle|\; \Gamma \text{ is polar to the curve } C\right\}} \subset \mathrm{Hilb}_6\left(\check{\mathbb{P}}^2\right).$$

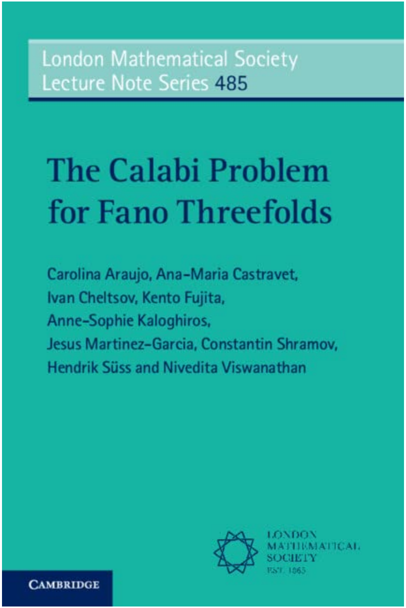
Then X is a smooth Fano 3-fold such that

- $(-K_X)^3 = 22$,
- $\mathrm{Pic}(X) \simeq \mathbb{Z}[-K_X]$,
- $\mathrm{Aut}(X) \simeq \mathrm{PSL}_2(\mathbf{F}_7)$.

X is K-stable, so general 3-fold in the family №1.10 is K-stable.

Conjectural description of all K-polystable members is in the **book**.

The Calabi problem for Fano threefolds, 2023, 450 pages



Smooth Fano 3-folds in the family №2.5

Let X be a smooth Fano threefold in the deformation family №2.5. Then there exists the following Sarkisov link:

$$\begin{array}{ccc} & X & \\ {}^{\pi}\swarrow & & \searrow^{\phi} \\ V & & \mathbb{P}^1 \end{array}$$

where

- V is a smooth cubic 3-fold in $\mathbb{P}^4$,
- π is a blow up of a smooth plane cubic curve,
- ϕ is a fibration into cubic surfaces.

Let P be a point in X, let S be a fiber of ϕ such that $P \in S$. Then S is a cubic surface in $\mathbb{P}^3$, and

- either S is a cone over a smooth cubic curve
- or S has Du Val singularities.

Lemma (C., Denisova, Fujita)

$\delta_P(X) > 1$ unless S has singularity of type $\mathbb{D}_5$ or $\mathbb{E}_6$ at P.

Smooth Fano 3-folds in the family №2.9

- Let $f = x^2z + y^2x + z^2t + t^2y$.
- Let $g = t^2x + tyz - x^2y + z^3$.
- Let $h = txy + xz^2 + y^2z - t^3$.
- Let $C = \{f = 0, g = 0, h = 0\} \subset \mathbb{P}^3$.

Then C is a smooth irreducible curve of genus 5 and degree 7.

```
Q:=RationalField();
P<x,y,z,t>:=ProjectiveSpace(Q,3);
X:=Scheme(P,[x^2*z+y^2*x+z^2*t+t^2*y,
             t^2*x+t*y*z-x^2*y+z^3,t*x*y+x*z^2+y^2*z-t^3]);
Degree(X);
IsNonsingular(X);
C:=Curve(X);
Genus(C);
```

Let $\pi\colon X \to \mathbb{P}^3$ be the blow up of the curve C.
Then X is a smooth Fano 3-fold in the family №2.9.
The 3-fold X is K-stable, so general 3-fold in this family is K-stable.
We expect that all smooth 3-folds in this family are K-stable.

Smooth Fano 3-folds in the family №2.10

Let V be the complete intersection of two quadrics in $\mathbb{P}^5$ given by

$$\begin{cases} x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0, \\ x_0^2 - x_1^2 + 2x_2^2 - 2x_3^2 + 3x_4^2 - 3x_5^2 = 0. \end{cases}$$

Let $C = \{x_0 = 0, x_1 = 0\} \cap V$. Then C is a smooth elliptic curve.

- Let $\pi\colon X \to V$ be the blow up of the curve C.
- Then X is a smooth Fano 3-fold in the family №2.10.

We have the following Sarkisov link:

$$\begin{array}{ccc} & X & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\phi} \\ V & & \mathbb{P}^1 \end{array}$$

where ϕ is a fibration into del Pezzo surfaces of degree 4.
X is K-stable, so general 3-fold in family №2.10 is K-stable.
We expect that all smooth 3-folds in this family are K-stable.

Smooth Fano 3-folds in the family №2.11

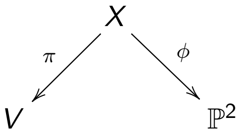
Let V be the cubic 3-fold in $\mathbb{P}^4$ given by

$$x_0x_3^2 + 2x_1x_3x_4 + x_2x_4^2 + 2x_2^2x_3 + 2x_0^2x_4 + ax_1^3 + bx_0x_1x_2 = 0.$$

where a and b are general numbers such that V is smooth.
For instance, we can set $a = 5$ and $b = 7$.

- Let L be the line $\{x_0 = x_1 = x_2 = 0\}$.
- Let $\pi\colon X \to V$ be the blow-up of the line L.
- Then X is a Fano 3-fold in the family №2.11.

We have the following Sarkisov link:



where ϕ is a conic bundle.
X is K-stable, so general 3-fold in family №2.11 is K-stable.
We expect that all smooth 3-folds in this family are K-stable.

Smooth Fano 3-folds in the family №2.12

Let X be a smooth complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$ given by

$$(x_0, x_1, x_2, x_3)M_1 \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = (x_0, x_1, x_2, x_3)M_2 \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = (x_0, x_1, x_2, x_3)M_3 \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0$$

where M_1, M_2, M_3 are 4×4 matrices.

Then X is a Fano 3-fold in the family №2.12.

Lemma (C., Li, Ma'u, Pinardin)

If $\mathrm{Aut}(X)$ is neither cyclic nor dihedral, then X is K-stable.

Example (Edge)

Let

$$X = \Big\{ x_0y_1 + x_1y_0 - \sqrt{2}x_2y_2 = x_0y_2 + x_2y_0 - \sqrt{2}x_3y_3 = x_0y_3 + x_3y_0 - \sqrt{2}x_1y_1 = 0 \Big\}.$$

Then X is smooth, and $\mathrm{Aut}(X) \simeq \mathrm{PSL}_2(\mathbb{F}_7) \times (\mathbb{Z}/2\mathbb{Z})$.

Hence, a general member of the family №2.12 is K-stable.

We expect that every smooth Fano 3-fold in this family is K-stable.

Smooth Fano 3-folds in the family №2.13

- Let $G = 2.\mathfrak{S}_4 \simeq \mathrm{GL}_2(\mathbf{F}_3)$ (GAP ID is [48,29]).
- Let C be the smooth genus 2 curve with a faithful G-action.
- Let $C \hookrightarrow \mathbb{P}^4$ be the G-equivariant embedding given by $|3K_C|$.
- Let Q be a G-invariant smooth quadric in $\mathbb{P}^4$ containing C.
- Let $\pi\colon X \to Q$ be the blow up of the curve C.
- Then X is a smooth Fano 3-fold in the family № 2.13.

We have the following G-Sarkisov link:

$$Q \xleftarrow{\ \pi\ } X \xrightarrow{\ \phi\ } \mathbb{P}^2$$

where ϕ is a conic bundle.

X is K-stable, so general 3-fold in family №2.13 is K-stable.

We expect that all smooth 3-folds in this family are K-stable.

Smooth Fano 3-folds in the family №2.14

- Let $V \subset \mathbb{P}^6$ be the smooth del Pezzo 3-fold of degree 5.
- Then $\mathrm{Aut}(V) \simeq \mathrm{PGL}_2(\mathbb{C})$.
- Let G be a subgroup in $\mathrm{Aut}(V)$ such that $G \simeq \mathrm{D}_{10}$.
- Then $\mathbb{P}^6$ has G-invariant 4-dimensional subspace Π.
- Set $C = V \cap \Pi$. Then C is a smooth elliptic curve.
- Let $\pi\colon X \to V$ be the blow up of the curve C.
- Then X is a smooth Fano 3-fold in the family №2.14.

We have the following G-Sarkisov link:

$$\begin{array}{ccc} & X & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\phi} \\ V & & \mathbb{P}^1 \end{array}$$

where ϕ is fibration into del Pezzo surfaces of degree 5.
X is K-stable, so general 3-fold in family №2.14 is K-stable.
We expect that all smooth 3-folds in this family are K-stable.

Smooth Fano 3-folds in the family №2.16

- Let V be a smooth intersection of two quadrics in $\mathbb{P}^4$.
- Let C be a smooth conic in V.
- Let $\pi\colon X \to V_4$ be the blow up of the conic C.
- Then X is a smooth Fano 3-fold in the family №2.16.

We have the following Sarkisov link:

$$\begin{array}{ccccc} & & X & & \\ & \overset{\pi}{\swarrow} & & \overset{\phi}{\searrow} & \\ V & & & & \mathbb{P}^2 \end{array}$$

where ϕ is a conic bundle.

Lemma (C., Hwang, Küronya, Laface, Mangolte, Massarenti, Park, Zhao, Zikas)

Let Δ_5 be the discriminant curve of the conic bundle ϕ.
If Δ_5 is smooth, then X is K-stable.

We expect that all smooth 3-folds in the family №2.16 are K-stable.

Smooth Fano 3-folds in the family №2.17

- Let C be be a smooth quintic elliptic curve in $\mathbb{P}^3$.
- Let $\pi\colon X \to \mathbb{P}^3$ be the blow up of this curve.
- Then X is a smooth Fano 3-fold in the family №2.17.

Lemma (C.,Pokora)

Suppose that $\mathrm{Aut}(X)$ has a subgroup isomorphic to $\mathbb{Z}/5\mathbb{Z}$.
Then X is K-stable.

Example

Let E be the harmonic elliptic curve, and let P be a point in E.
Then

$$\mathrm{Aut}\big(E, [5P]\big) \simeq \big(\mathbb{Z}/5\mathbb{Z}\big) \rtimes \big(\mathbb{Z}/4\mathbb{Z}\big),$$

and there is $\mathrm{Aut}\big(E, [5P]\big)$-equivariant embedding $E \hookrightarrow \mathbb{P}^4$ whose image is a smooth quintic elliptic curve. If C this curve, then

$$\mathrm{Aut}(X) \simeq \mathrm{Aut}\big(E, [5P]\big) \simeq \big(\mathbb{Z}/5\mathbb{Z}\big) \rtimes \big(\mathbb{Z}/4\mathbb{Z}\big).$$

We expect that all smooth 3-folds in the family №2.17 are K-stable.

Smooth Fano 3-folds in the family №2.20

- Let $Q = \{xt = yz + w^2\} \subset \mathbb{P}^4$.
- Let $\chi\colon Q \dashrightarrow \mathbb{P}^6$ be the rational map given by
$$\big[x : y : z : t : w\big] \mapsto \big[wx : wy : wz : wt : w^2 : xz-y^2 : yt-z^2\big].$$
- Let V be the closure of the image of χ.
- Then V is a smooth del Pezzo 3-fold of degree 5.

Let C be the twisted cubic in V given parametrically as
$$[r^3 : r^2s : rs^2 : s^3 : 0 : 0 : 0],$$
and let $\pi\colon X \to V$ be the blow up of the curve C. Then
$$\mathrm{Aut}(X) \simeq \mathbb{C}^* \rtimes \big(\mathbb{Z}/2\mathbb{Z}\big),$$
and X is the unique smooth Fano 3-fold in the family №2.20 that has an infinite automorphism group.

Lemma (Araujo, Castravet, C., Fujita, Kaloghiros, Martinez-Garcia, Shramov, Süß, Viswanathan)

The 3-fold X is K-polystable.

This implies that general member of the family №2.20 is K-stable.

Smooth Fano 3-folds in the family №3.2

Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, let H be the divisor of degree $(1, 1)$ on S, let

$$\mathbb{P} = \mathbb{P}\big(\mathcal{O}_S \oplus \mathcal{O}_S(-H) \oplus \mathcal{O}_S(-H)\big),$$

let $[s_0 : s_1; t_0 : t_1; u_0 : u_1 : u_2]$ be coordinates on $\mathbb{P}$ such that

$$\mathrm{wt}(s_0) = (1,0,0),\ \mathrm{wt}(s_1) = (1,0,0),\ \mathrm{wt}(t_0) = (0,1,0),\ \mathrm{wt}(t_1) = (0,1,0),$$
$$\mathrm{wt}(u_0) = (0,0,1),\ \mathrm{wt}(u_1) = (1,1,1) \text{ and } \mathrm{wt}(u_2) = (1,1,1).$$

- Let L be the tautological line bundle on $\mathbb{P}$ over S.
- Let X be the 3-fold in $|L^{\otimes 2} \otimes \mathcal{O}_S(2,3)|$ given by

$$t_0u_1^2+t_1u_2^2+u_0\big(s_0t_0^2u_1+s_1t_1^2u_2+s_0t_1^2u_1+s_1t_0^2u_2\big)+u_0^2\big(s_0^2t_0^3+s_1^2t_1^3+s_0^2t_0t_1^2+s_1^2t_0^2t_1\big)=0.$$

- Then X is a smooth Fano threefold in the family №3.2.

Lemma (Araujo, Castravet, C., Fujita, Kaloghiros, Martinez-Garcia, Shramov, Süß, Viswanathan)

The 3-fold X is K-stable.

Thus, general member of the family №3.2 is also K-stable.
We expect that all smooth members of this family are K-stable.

Smooth Fano 3-folds in the family №3.5

Let $S = \mathbb{P}^1_{u,v} \times \mathbb{P}^1_{x,y}$, and let $C \subset S$ be a smooth curve given by

$$u\big(x^5+a_1x^4y+a_2x^3y^2+a_3x^2y^3\big) = v\big(y^5+b_1xy^4+b_2x^2y^3+b_3x^3y^2\big)$$

Consider the embedding $S \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ given by

$$\big([u:v],[x:y]\big) \mapsto \big([u:v],[x^2:xy:y^2]\big),$$

and identify S and C with their images in $\mathbb{P}^1 \times \mathbb{P}^2$.

- Let $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ be the blow up of the curve C.
- Then X is a smooth Fano 3-fold in the family № 3.5.

Theorem (Denisova)

Let $\eta\colon C \to \mathbb{P}^1_{u,v}$ be the projection $([u:v],[x:y]) \mapsto [u:v])$.
If all ramification points of η have index 2, then X is K-stable.

We expect that X is K-polystable $\iff$ C is GIT-polystable.

Smooth Fano 3-folds in the family №3.6

- Let L be a line in $\mathbb{P}^3$.
- Let C be a quartic elliptic curve in $\mathbb{P}^3 \setminus L$.
- Let $\pi\colon X \to \mathbb{P}^3$ be the blow up of L and C.
- Then X is a smooth Fano 3-fold in the family №3.6.

We have commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\pi}\swarrow & & \searrow^{\sigma} \\ \mathbb{P}^3 & \overset{\varsigma}{\dashrightarrow} & \mathbb{P}^1 \end{array}$$

where

- ς is the projection from L,
- σ is a fibration into del Pezzo surfaces of degree 5.

Lemma (C.)

Suppose that every singular fiber of σ has one singular point, and this point is a singular point of type $\mathbb{A}_1$ or of type $\mathbb{A}_2$. Then X is K-stable.

We expect that all smooth members of this family are K-stable.

Smooth Fano 3-folds in the family №3.7

Let $V = \{x_0y_0 + x_1y_1 + x_2y_2 = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^2$, and let

$$C = \left\{x_0y_1 + \omega x_1y_2 + \omega^2 x_2y_0 = x_0y_2 + \omega x_1y_0 + \omega^2 x_2y_1 = 0\right\} \cap V,$$

where ω is a primitive cube root of unity.
Then V is the unique smooth Fano 3-fold in the family 2.32.
Note that C is a smooth elliptic curve in V.

- Let $\pi\colon X \to V$ be the blow up of the curve C.
- Then X is a smooth Fano 3-fold in the family №3.7.

We have the following Sarkisov link:

$$\begin{array}{ccc} & X & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\phi} \\ V & & \mathbb{P}^1 \end{array}$$

where ϕ is fibration into del Pezzo surfaces of degree 6.
X is K-stable, so general 3-fold in family №3.7 is K-stable.
We expect that all smooth 3-folds in this family are K-stable.

Smooth Fano 3-folds in the family №3.8

Let $S = \mathbb{P}^1_{u,v} \times \mathbb{P}^1_{x,y}$, and let $C \subset S$ be a smooth curve given by

$$u\left(x^4 + a_1x^3y + a_2x^2y^2\right) + v\left(y^4 + b_1y^3x + b_2y^2x^2\right) = 0.$$

Consider the embedding $S \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ given by

$$\left([u:v],[x:y]\right) \mapsto \left([u:v],[x^2:xy:y^2]\right),$$

and identify S and C with their images in $\mathbb{P}^1 \times \mathbb{P}^2$.

- Let $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ be the blow up of the curve C.
- Then X is a smooth Fano 3-fold in the family № 3.8.

Theorem (Araujo, Castravet, C., Fujita, Kaloghiros, Martinez-Garcia, Shramov, Süß, Viswanathan)

Suppose $\mathrm{Aut}(S, C)$ does not leave invariant fibers of $S \to \mathbb{P}_{u,v}$. Then X is K-polystable.

Using this, it is easy to construct an example of a K-stable X. We expect that X is K-polystable $\iff$ C is GIT-polystable.

Smooth Fano 3-folds in the family №3.11

- Let C be a smooth quartic elliptic curve in $\mathbb{P}^3$.
- Let P be a point in C.
- Let $\phi\colon V \to \mathbb{P}^3$ be the blowup of a point P.
- Let $\widetilde{C}$ be the strict transform on V of the curve C.
- Let $\pi\colon X \to V$ be the blow up of the curve $\widetilde{C}$.
- Then X is a smooth Fano 3-folds in the family №3.11.

Theorem (Fujita)

Suppose that $P = [0 : 0 : 1 : 0]$ and

$$C = \{x^2 + y^2 + zt = yz + t^2 = 0\} \subset \mathbb{P}^3.$$

Then $\mathrm{Aut}(X) \simeq \mathbb{Z}/6\mathbb{Z}$ and X is K-stable.

Thus, general members of the family №3.11 are K-stable.

Remark

If $\mathrm{Aut}(X)$ is trivial, X is K-stable (Smiech, work in progress).

We expect that all smooth members in this family are K-stable.

K-polystable singular limits of smooth Fano 3-folds

All K-polystable singular limits of smooth Fano 3-folds in the deformation families

№1.12, №1.13, №1.14, №2.15, №2.15, №2.18, №2.19, №2.21, №2.22,
№2.24, №2.25, №3.9, №3.10, №3.12, №3.13, №4.1, №4.2, №4.13

have been found (Abban, Ascher, C., Denisova, DeVleming, Etxabarri-Alberdi, Fedorchuk, Ji, Jiao, Kaloghiros, Kennedy-Hunt, Liu, Martinez-Garcia, Papazachariou, Quek, Spotti, Sun, Thompson, Xu, Zhao).

For instance, there are 8 deformation families of smooth Fano 3-folds that have one-dimensional moduli:

№2.22, №2.24, №2.25, №2.28, №3.12, №3.13, №3.14, №4.13.

But only 6 of them have K-polystable members:

№2.22, №2.24, №2.25, №3.12, №3.13, №4.13.

K-polystable limits of smooth Fano 3-folds in these families are found by Abban, C., Denisova, Etxabarri-Alberdi, Kaloghiros, Jiao, Martinez-Garcia, Papazachariou.

Two-dimensional components of the K-moduli space

There are 6 families with two-dimensional moduli:

№2.23, №2.21, №3.10, №3.11, №4.2, №7.1.

But only 5 of them have K-polystable smooth members:

- **№2.21** blow up of a quadric 3-fold in twisted quartic curve;
- **№3.10** blow up of a quadric 3-fold in two conics;
- **№3.11** blow up of $\mathbb{P}^3$ at a point followed by the blow up of a strict transform of a smooth elliptic curve of degree 4 that passes through this point;
- **№4.2** blow up of a quadric cone in $\mathbb{P}^4$ with one singular point at the vertex and a smooth elliptic curve of degree 4;
- **№7.1** $\mathbb{P}^1 \times S$, where S is a smooth del Pezzo surface of degree 4.

K-polystable limits of smooth 3-folds in the families №2.21, №3.10, №4.2, №7.1 are explicitly found (C., Guerreiro, Fujita, Krylov, Mabuchi, Malbon, Martinez-Garcia, Mukai, Thompson).

Johnson–Kollár 3-folds

Let $X \subset \mathbb{P}(a_1, a_2, a_3, a_4, a_5)$ be a well-formed quasismooth hypersurface of degree $d = a_1 + a_2 + a_3 + a_4 + a_5 - 1$.
Johnson and Kollár showed that either there is an odd k such that

$$\big(a_1, a_2, a_3, a_4, a_5, d\big) = \big(2, kb_1, kb_2, kb_3, k(b_1 + b_2 + b_3) - 1\big),$$

where (b_1, b_2, b_3) is one of the following 25 triples:

$$(1,1,1), (1,1,2), (1,1,3), (1,1,4), (1,2,3), (1,2,5), (1,3,4), (1,3,5),$$
$$(1,3,7), (1,3,8), (1,4,5), (1,4,9), (1,5,7), (1,5,12), (2,3,5), (2,3,7), (2,5,9),$$
$$(3,4,5), (3,4,7), (3,4,11), (3,5,11), (3,5,16), (4,5,7), (4,5,13), (5,7,8),$$

or $(a_1, a_2, a_3, a_4, a_5, d)$ is one of the 4442 sporadic sixtuples.

Theorem (Johnson, Kollár)

If $(a_1, a_2, a_3, a_4, a_5, d)$ is contained in 1936 out of 4442 sporadic sixtuples, then X is K-stable.

Theorem (Campo, C., Kim, Okada, Sano, Tasin, Won)

If X has terminal singularities, then X is K-stable.

We expect that the Fano 3-fold X is always K-stable.

Mukai varieties of Picard rank 1

Let X be a smooth Fano variety of dimension n such that

$$-K_X \sim (n-2)H,$$

where $\mathrm{Pic}(X) = \mathbb{Z}[H]$. Let $g \in \mathbb{N}$ such that $2g - 2 = H^n$.

- Suppose that X is not a weighted complete intersection.

Then X is a linear section of one of the following varieties:

($g = 6$) intersection of the cone over $\mathrm{Gr}(2,5)$ with a quadric in $\mathbb{P}^{10}$,

($g = 7$) $\mathrm{OGr}_+(5,10) \subset \mathbb{P}^{15}$, a connected component of the Grassmannian of isotropic 5-dimensional subspaces in a 10-dimensional space equipped with a quadratic form,

($g = 8$) $\mathrm{Gr}(2,6) \subset \mathbb{P}^{14}$,

($g = 9$) $\mathrm{LGr}(3,6) \subset \mathbb{P}^{13}$, the Grassmannian of Lagrangian 3-dimensional subspaces in a 6-dimensional symplectic space,

($g = 10$) $\mathrm{G_2Gr}(2,7) \subset \mathbb{P}^{13}$, the adjoint Grassmannian of the group G_2,

($g = 12$) $\mathrm{I_3Gr}(3,7) \subset \mathbb{P}^{13}$, the Grassmannian of 3-dimensional subspaces in a 7-dimensional vector space isotropic for a (general) triple of skew symmetric forms.

If X is maximal, then $n = 5, 10, 8, 6, 5, 3$, respectively.

Maximal Mukai varieties of Picard rank $\geqslant 1$

($g = 7$) Double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ branched over $(2,2)$-divisor.
($g = 9$) Divisor in $\mathbb{P}^2 \times \mathbb{P}^3$ of degree $(1,2)$.
($g = 11$) $\mathbb{P}^3 \times \mathbb{P}^3$.
($g = 11$) $\mathbb{P}^3 \times Q$, where Q is a quadric 3-fold.
($g = 12$) Blow up of quadric 4-fold along conic.
($g = 13$) Flag variety of $\mathrm{SO}_5(\mathbb{C})$.
($g = 14$) Blow up of $\mathbb{P}^5$ along a line.
($g = 16$) $\mathbb{P}\big(\mathcal{O}_Q \oplus \mathcal{O}_Q(1)\big)$, where Q is a quadric 3-fold.
($g = 21$) $\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2)\big)$.

K-stability of Mukai varieties are almost not studied.

Example

There is a one-dimensional family of Mukai fourfolds of genus ten:

1. one with $\mathrm{Aut}(X) \simeq \mathrm{GL}_2(\mathbb{C}) \rtimes \big(\mathbb{Z}/2\mathbb{Z}\big)$,
2. one with $\mathrm{Aut}(X) \simeq \big(\mathbb{C}_+ \times \mathbb{C}^*\big) \rtimes \big(\mathbb{Z}/2\mathbb{Z}\big)$,
3. others have $\mathrm{Aut}(X) \simeq \big(\mathbb{C}^*\big)^2 \rtimes \big(\mathbb{Z}/6\mathbb{Z}\big)$.

The first one is K-polystable (Fujita).