K-moduli of Fano threefolds via moduli continuity method

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Simons Conference on Moduli of Varieties

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January 17, 2025

K-moduli theorem

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Theorem (K-moduli theorem; Alper, Blum, Halpern-Leistner, Jiang, Li, Liu, Wang, Xu, Zhuang, etc.)

Fix a positive integer n and a rational number $V > 0$. Consider the moduli pseudo-functor sending a base scheme S to

$$\left\{ \mathscr{X}/S \;\middle|\; \begin{array}{l} \mathscr{X}/S \text{ is a family of } \mathbb{Q}\text{-Fano varieties,} \\ \text{each fiber } \mathscr{X}_s \text{ is K-semistable, and} \\ \dim \mathscr{X}_t = n \text{ and } (-K_{\mathscr{X}_t})^n = V. \end{array} \right\}.$$

Then there is an Artin stack, denoted by $\mathcal{M}^K_{n,V}$, of finite type over $\mathbb{C}$ which represents the pseudo-functor.

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Theorem (K-moduli theorem cont.)

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- *The* $\mathbb{C}$*-points of* $\overline{M}^K_{n,V}$ *parameterize* n*-dimensional K-polystable* $\mathbb{Q}$*-Fano varieties* X *of volume* V*.*
- *The Chow-Mumford (abbv. CM)* $\mathbb{Q}$*-line bundle* λ_{CM} *on* $\mathcal{M}^K_{n,V}$ *descends to an ample* $\mathbb{Q}$*-line bundle* Λ_{CM} *on* $\overline{M}^K_{n,V}$*.*

Question

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Goal

Moduli continuity method, with two highlighted examples of Fano threefolds.

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This is usually done by equivariant method and δ-invariant estimates.

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Theorem (Liu'16)

Let X be an n-dimensional K-semistable Fano variety, and $x \in X$ a closed point. Then

$$\frac{\mathrm{vol}(X)}{\mathrm{vol}(\mathbb{P}^n)} \leq \frac{\widehat{\mathrm{vol}}(x, X)}{\widehat{\mathrm{vol}}(0, \mathbb{P}^n)}.$$

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In particular, X is Gorenstein canonical if the inequality holds.

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Goal

- Show that varieties on the boundary share similar properties with generic one.
- Put all $X \in \mathcal{M}^K$ in a suitable parameter space W.

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Compute λ_{CM} on $\mathscr{X} \to W$, and then identify

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- This is usually where the properness of $\overline{M}^K$ comes in.

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- Compute CM line bundle on $\mathbb{P}\mathcal{E}$: $N_t = \xi + t\eta$.

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Theorem (Liu–Z.'24)

Let $t_0 = \frac{22}{51}$. Then one has

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where the latter is the birational model of $\overline{M}_4$ *in Hassett-Keel program for any* $\alpha_0 \in (\frac{1}{2}, \frac{23}{44})$.

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Remark

The K-stability of smooth Fano №2.15 is proved by Duarte-Guerreiro–Giovenzana–Viswanathan using *Abban–Zhuang's method*.

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Proposition (Liu–Z'24)

Any $X \in \mathcal{M}^K_{\text{№}2.15}$ *admits a Sarkisov link structure*

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  (4) $\widetilde{L} := \widetilde{\mathcal{L}}|_{\widetilde{X}}$, $\widetilde{Q} := \widetilde{\mathscr{Q}}|_{\widetilde{X}}$, $\widetilde{H} := \widetilde{\mathcal{H}}|_{\widetilde{X}}$, and $\widetilde{E} := \widetilde{\mathscr{E}}|_{\widetilde{X}}$ are all Cartier.

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- Idea: take $S \in |-K_{\widetilde{X}}|$ a general elephant (K3 surface); first show $\widetilde{L}|_S$ is nef, then lift sections.

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- $(S_t, 2L_t - (1-\epsilon)Q_t) \rightsquigarrow (S_0, 2L_0 - (1-\epsilon)Q_0)$ as degeneration of polarized K3 surfaces.

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- (Moduli of lattice-polarized K3 surfaces) If $S'_t \rightsquigarrow S'_0$ as degeneration of degree 6 K3, then $S_0 \simeq \mathrm{Bl}_{p_0} S'_0$ and hence $\widetilde{L}|_S$ is nef (as the pullback of L_0 from S_0).

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- Thus Bs$|2\widetilde{L}|$ = points $\cup$ g-exceptional subsets, where $g : \widetilde{X} \to X$ is the birational modification.
- Since $\widetilde{L} = \frac{1}{2}(-K_{\widetilde{X}} + Q)$ is g-ample, it is nef.

Moduli of vector bundles and bundle stable pairs

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- (Thaddeus'94) Let $\sigma \in \mathbb{Q}_{>0}$ and $\overline{N}_C(\sigma, \Lambda)$ be the moduli space of σ-semistable pairs (E, ϕ), where E is a rank 2 vector bundle with determinant Λ and $\phi \in H^0(C, E)$.

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- $\overline{N}_C(\sigma, \Lambda)$ is non-empty if and only if $\sigma \leq d - \frac{1}{2}$

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- (Boundary model I) There is a natural isomorphism

$$\overline{N}_{C,0}(\Lambda) \ \simeq\ \mathbb{P}H^0(K_C \otimes \Lambda)^* \ \simeq\ \mathbb{P}^{3g-3},$$

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- (Intermediate models) There is a divisorial contraction $\phi : \overline{N}_{C,1}(\Lambda) \to \overline{N}_{C,0}(\Lambda)$ which blows up $\overline{N}_{C,0}(\Lambda)$ along C. All the other intermediate moduli are connected by flips.

Wall crossing

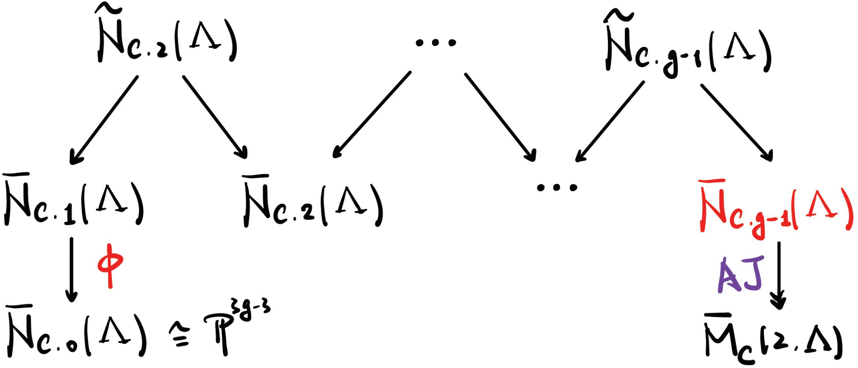

Figure: Wall crossing of Thaddeus' moduli of stable pairs

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- The wall-crossing structure coincides with Sarkisov link of Fano family №2.19.

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- Is there a natural forgetful morphism from the K-moduli of $\overline{N}_C(2, \Lambda)$ to that of $\overline{M}_C(2, \Lambda)$?

Acknowledgement

Thanks