The volume preserving Cremona group

Zhijia Zhang, New York University

joint with Konstantin Loginov

January 18, 2025, UCSD

$\mathrm{char}(k) = 0$.

Introduction

Let k be a field of characteristic zero, and ω_n be the standard torus-invariant volume form on $\mathbb{P}^n$ with logarithmic poles, given (in affine chart) by

$$\omega_n = \frac{\mathrm{d}x_1}{x_1} \wedge \ldots \wedge \frac{\mathrm{d}x_n}{x_n}.$$

We are interested in the group of birational automorphisms of $\mathbb{P}^n$ over k preserving ω_n:

$$\mathrm{Bir}_k(\mathbb{P}^n, \omega_n) := \{\varphi \in \mathrm{Cr}_n(k) : \varphi^*(\omega_n) = \omega_n\}.$$

This is a subgroup of the Cremona group $\mathrm{Cr}_n(k)$, the group of birational automorphisms of $\mathbb{P}^n$ over k.

The Cremona group

Theorem (Noether 1870, Castelnuovo 1901)

Over an algebraically closed field k, the Cremona group $\mathrm{Cr}_2(k)$ is generated by $\mathrm{PGL}_3(k)$ and the standard Cremona involution

$$(x : y : z) \mapsto (\frac{1}{x} : \frac{1}{y} : \frac{1}{z}).$$

Note that this is not true if k is not algebraically closed, e.g., $k = \mathbb{R}$.

Theorem (Usnich 2006, Blanc 2013)

Over an algebraically closed field k, the group $\mathrm{Bir}_k(\mathbb{P}^2, \omega_2)$ is generated by $\mathbb{G}_m^2, \mathrm{SL}_2(\mathbb{Z})$ and the cluster transformation of order 5

$$(x, y) \mapsto (y, \frac{y+1}{x}).$$

The Cremona group

Challenge: finding (explicit) generators for $\mathrm{Cr}_n(k)$ for $n \geq 3$ is still open.

Theorem (Lin-Shinder 2024)

In each of the following cases, $\mathrm{Cr}_n(k)$ is not generated by **pseudo-regularizable** elements (which include any birational map of finite order and automorphisms):

1. $n \geq 3$ and k is a number field; or the function field of an algebraic variety over a number field, over a finite field, or over an algebraically closed field,
2. $n \geq 4$ and k is a subfield of $\mathbb{C}$,
3. $n \geq 5$ and k is any infinite field.

Motivic invariant of birational maps

Definition. A birational map $\varphi : X \dashrightarrow X$ is called pseudo-regularizable if $\varphi = \alpha^{-1} \circ \beta \circ \alpha$ where

- $\alpha : X \dashrightarrow Y$ is a birational map and Y is a projective variety,
- $\beta : Y \dashrightarrow Y$ is an isomorphism in codimension 1.

Motivic invariant. Given a birational map $\varphi : X \dashrightarrow Y$, let

$$c(\varphi) = \sum_i [E_i] - \sum_j [F_j] \in \mathrm{Burn}_{n-1}(k)$$

where the sum of E_i (resp. F_i) runs over k-irreducible components of the exceptional divisor of φ (resp. φ^{-1}), and

$$\mathrm{Burn}_{n-1}(k) = \mathbb{Z} \cdot \begin{bmatrix} \text{birational isomorphism} \\ \text{classes of varieties of} \\ \text{dimension } n-1 \text{ over } k \end{bmatrix}.$$

Theorem (Lin-Shinder 2024)

$c(\varphi \circ \psi) = c(\varphi) + c(\psi)$.

Thus $c(\text{psuedo-regularizable map}) = 0$.

Under their assumptions on n and k, there exists $\varphi \in \mathrm{Cr}_n(k)$ s.t. $c(\varphi) \neq 0$. Then $\mathrm{Cr}_n(k)$ is not generated by pseudo-regularizable maps.

Theorem (Loginov-Z. 2024)

In each of the following cases, $\mathrm{Bir}_k(\mathbb{P}^n, \omega_n)$ is not generated by pseudo-regularizable elements, and is **not simple**:

1. $n \geq 3$ and k is a number field; or the function field of an algebraic variety over a number field, over a finite field, or over an algebraically closed field,
2. $n \geq 4$ and $k = \mathbb{C}$.

Log Calabi-Yau pairs

Corollary (Loginov-Z. 2024)

Under the same assumption, the same holds for

$$\mathrm{Bir}_k(\mathbb{P}^n, \Delta_n)$$

the group of crepant birational automorphisms of the toric log Calabi-Yau pairs $(\mathbb{P}^n, \Delta_n)$, where $\Delta_n = \sum_{i=1}^{n+1}\{x_i = 0\}$ is the sum of coordinate hyperplanes.

$$\left\{\begin{array}{l}\text{crepant birational maps}\\ \text{of log Calabi-Yau pairs}\end{array}\right\} \sim \left\{\begin{array}{l}\text{birational maps preserving}\\ \text{volume forms (up to scalar)}\end{array}\right\}$$

Goal/Proof: construct $\varphi \in \mathrm{Bir}_k(\mathbb{P}^n, \Delta_n)$ such that $c(\varphi) \neq 0$.

Dimension 4: K3 surface

Let $k = \mathbb{C}$. The following construction is due to Hassett-Lai:

Let R_L be a K3 surface of degree 12, with polarization $\Gamma^2 = 12$. Γ gives an embedding $R_L \hookrightarrow \mathbb{P}^7$. Pick three general points $x_1, x_2, x_3 \in R_L$ and project from it:

$$R_L \dashrightarrow S_L \subset \mathbb{P}^4.$$

S_L is a non-normal surface, with three transverse double points p_1, p_2, p_3. The linear system $|\mathcal{O}_{\mathbb{P}^4}(4) - S_L|$ gives rise to a birational map in $\mathrm{Cr}_4(\mathbb{C})$

$$\begin{array}{ccc} & X & \\ {\scriptstyle \pi_L = \mathrm{Bl}_{S_L}} \swarrow & & \searrow {\scriptstyle \pi_M = \mathrm{Bl}_{S_M}} \\ S_L \subset \mathbb{P}^4 & \overset{\varphi}{\dashrightarrow} & \mathbb{P}^4 \supset S_M \end{array}$$

where S_M is obtained in the same way from projection from another K3 of degree 12. We have

Theorem (Hassett-Lai 2018)

For a **general** S_L arising from this construction, one has $\mathrm{Pic}(S_L) = \mathbb{Z}$, S_L is **not** isomorphic to S_M, and S_L is **derived equivalent** to S_M.

Proof: Construction of an explicit example and computation of

$$H^4(X, \mathbb{Z})_{\mathrm{alg}} = \langle L^2, \widetilde{\Gamma}_L, Q_1, Q_2, Q_3, \widetilde{F}_i \rangle = \langle M^2, \widetilde{\Gamma}_M, K_1, K_2, K_3, \widetilde{G}_i \rangle$$

where

- L: pullback of a general hyperplane section on $\mathbb{P}^4$,
- $\widetilde{\Gamma}_L$: preimage (in X) of the image (in $\mathbb{P}^4$) of the polarization Γ_L,
- Q_i: quadric surface in X above the singular point $p_i \in S_L$,
- $\widetilde{F}_i$: preimage in X of the (-1)-curves in S_L (arised from the projection).
- And the "mirrored" objects coming from the K3 on the other

Goal: find quintic on $\mathbb{P}^4$ such that φ extends to a crepant birational map:

$$\begin{array}{ccc} & X & \\ {}^{\pi_L}\swarrow & & \searrow^{\pi_M} \\ (S_L \subset \mathbb{P}^4, ?H_L + ?H_L + B_L) & \overset{\varphi}{\dashrightarrow} & (S_M \subset \mathbb{P}^4, ?H_M + ?H_M + B_M) \end{array}$$

Construction:

1. Pick singular points $p_1 \in S_L$, $q_1 \in S_M$. Consider Q_1, K_1 the quadric surface in X above them.
   Let $H_L = \mathbb{P}^3 \supset \pi_L(K_1)$ and $H_M = \mathbb{P}^3 \supset \pi_M(Q_1)$.
2. Let $B_L = \varphi_*^{-1}(H_M)$ and $B_M = \varphi_*(H_L)$.Then B_L, B_M are quartic threefolds (with singularities).

Using intersection theory: $\pi_L(K_1) \simeq \pi_M(Q_1) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. And

$$\begin{aligned} B_L \cap H_L &= \operatorname{span}(p_1, p_2, p_3) \cup \operatorname{span}(F_1, p_1) \cup \pi_L(K_1) \\ &= \mathbb{P}^2 \cup \mathbb{P}^2 \cup \mathbb{P}^1 \times \mathbb{P}^1. \end{aligned}$$

Using these observations, we construct the diagram

$$\begin{array}{ccc} (S_L \subset \mathbb{P}^4, H_L + B_L) & \overset{\varphi}{\dashrightarrow} & (S_M \subset \mathbb{P}^4, H_M + B_M) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ (\mathbb{P}^4, \Delta_4) & \overset{\varphi'}{\dashrightarrow} & (\mathbb{P}^4, \Delta_4) \end{array}$$

where f, g are crepant birational maps with $c(f) = c(g) = 0$, and $\varphi' \in \operatorname{Bir}_{\mathbb{C}}(\mathbb{P}^4, \Delta_4)$ with $c(\varphi') \neq 0$.

Dimension 3: quintic genus one curve

Let C be a quintic genus one curve, $C = \mathrm{Gr}(2,5) \cap \mathbb{P}^4$.

C is also the base locus of $\mathcal{M} = |\mathcal{O}_{\mathbb{P}^4}(2) - C|$. Let Q be a general member in $\mathcal{M}$. Then $\mathcal{M}$ gives rise to a birational map φ with resolution

$$\begin{array}{ccc} & Z & \\ \swarrow^{\mathrm{Bl}_C} & & \searrow^{\mathrm{Bl}_{C'}} \\ C \subset Q & \overset{\varphi}{\dashrightarrow} & \mathbb{P}^3 \supset C' \end{array}$$

where $C' \simeq_k \mathrm{Jac}^2(C)$, which is also a quintic genus one curve. We have

$$c(\varphi) = [C \times \mathbb{P}^1] - [C' \times \mathbb{P}^1].$$

Theorem (Lin-Shinder) Under the aforementioned assumption on k, there exists such C with no k-points, and not k-isomorphic to C'.

Dimension 3: quintic genus one curve

Goal: find anti-canonical divisors on Q and $\mathbb{P}^3$ such that φ extends to a crepant birational map:

$$(Q, ?Q_1 + ?Q_1 + H_1) \overset{\varphi}{\dashrightarrow} (\mathbb{P}^3, ?H_2 + ?H_2 + S_1)$$

Construction:

1. Pick a general plane $H_2 = \mathbb{P}^2$ in $\mathbb{P}^3$, put $Q_1 = \varphi_*^{-1}(H_2)$. Then Q_1 is a smooth dP_4.
2. $H_2 \cap C' = 5$ points. Let R be the conic in H_2 passing through these 5 points. Then its strict transform $\varphi_*^{-1}(R)$ is a line ℓ in Q_1.
3. Pick a general point q on ℓ. Let H_1 be the tangent hyperplane section of Q at q, and put $S_1 = \varphi_*(H_1)$. Then H_1 is a quadric cone, S_1 is an A_1-cubic surface.

Dimension 3: quintic genus one curve

Goal: find anti-canonical divisors on Q and $\mathbb{P}^3$ such that φ extends to a crepant birational map:

$$(Q, Q_1 + H_1) \overset{\varphi}{\dashrightarrow} (\mathbb{P}^3, H_2 + S_1)$$

$$\begin{array}{ccc}
(Q, Q_1 + H_1) & \overset{\varphi}{\dashrightarrow} & (\mathbb{P}^3, H_2 + S_1) \\
{\scriptstyle f} \downarrow & & \downarrow {\scriptstyle g} \\
(\mathbb{P}^3, \Delta_3) & \overset{\varphi'}{\dashrightarrow} & (\mathbb{P}^3, \Delta_3)
\end{array}$$

- Q_1 =dP$_4$, $H_1 = \mathbb{P}(1,1,2)$. And $Q_1 \cap H_1$=a twisted cubic+a line intersecting at two points.
- $H_2 = \mathbb{P}^2$, S_1 = A$_1$-cubic surface. And $H_2 \cap S_1$=a conic+a line intersecting at two points.
- Then we construct crepant birational maps f, g such that

Birational geometry of log Calabi-Yau pairs

Let (X, D_X) be a log Calabi-Yau pair. The **coregularity** of (X, D_X) is the number $(\dim X - 1 - \dim(\text{the dual complex of } (X, D_X)))$.

Birational geometry of log CY pair with "large" corregularity is *rigid*:

Theorem (Araujo-Corti-Massenrenti 2023)

$\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^3, D) = \mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^3, D)$ for a general irreducible smooth quartic surface D. The appearance of singularities on D enriches the birational geometry of the pair.

The case (toric pairs) we studied has corregularity 0, and has opposite behaviour.

Thank you!