Pseudoeffective $\mathbb{R}$-divisors with volume zero on K-trivial manifolds

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K-trivial varieties and their moduli
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joint work with Simion Filip and John Lesieutre

Positive cones

X^n a smooth projective variety over $\mathbb{C}$. Real Néron-Severi group $N^1(X,\mathbb{R}) = N^1(X) \otimes \mathbb{R}$

Two open convex cones $\mathcal{A} \subset \mathcal{B} \subset N^1(X,\mathbb{R})$

$$\mathcal{A} = \left\{ \sum_i c_i A_i \;\middle|\; c_i \in \mathbb{R}_{>0}, A_i \text{ ample Cartier divisor} \right\}$$

$$\mathcal{B} = \left\{ \sum_i c_i B_i \;\middle|\; c_i \in \mathbb{R}_{>0}, B_i \text{ big Cartier divisor} \right\}$$

D Cartier divisor is called big if

$$\mathrm{Vol}(D) := \limsup_{m\to+\infty} \frac{h^0(X,\mathcal{O}_X(mD))}{m^n/n!} > 0$$

Volume function

$$\mathrm{Vol}(D) := \limsup_{m\to+\infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!} \in [0, +\infty)$$

D ample $\Rightarrow$ big, $\mathrm{Vol}(D) = \frac{(D^n)}{n!}$

Fujita: the limsup is actually a limit

If D is nef (i.e. $(D \cdot C) \geqslant 0$ for all curves $C \subset X$) then $\mathrm{Vol}(D) = \frac{(D^n)}{n!} \geqslant 0$

Lazarsfeld: If $D \equiv D'$ then $\mathrm{Vol}(D) = \mathrm{Vol}(D')$, so volume descends to $N^1(X)$

Volume is homogeneous of degree n, so it can be naturally extended to $\mathbb{Q}$-divisors.

Vol is a locally Lipschitz function on $N^1(X) \otimes \mathbb{Q}$, and it extends to a continuous function

$$\mathrm{Vol} : N^1(X, \mathbb{R}) \to \mathbb{R}_{\geqslant 0}$$

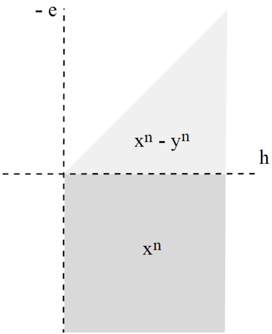
More positive cones

$$\mathcal{B} = \{D \in N^1(X,\mathbb{R}) \mid \mathrm{Vol}(D) > 0\}$$

$\overline{\mathcal{A}}$ =cone of nef $\mathbb{R}$-divisors

$\mathcal{E} := \overline{\mathcal{B}}$ =cone of pseudoeffective $\mathbb{R}$-divisors. $\mathrm{Vol}|_{\partial\mathcal{E}} \equiv 0$

Ex. $X = \mathrm{Bl}_p\mathbb{P}^n$, $n \geqslant 2$, $N^1(X,\mathbb{R}) \cong \mathbb{R}^2$ spanned by $h = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$, $e = \mathcal{O}_X(E)$



$$\mathcal{A} = \{xh - ye \mid 0 < y < x\}$$
$$\mathcal{B} = \{xh - ye \mid x > 0, y < x\}$$

$$\mathrm{Vol}(xh - ye) = \begin{cases} x^n - y^n, & \text{if } y \geqslant 0 \\ x^n, & \text{if } y < 0 \end{cases}$$

Regularity of Vol

Question (Lazarsfeld)

What is the regularity of Vol *in* $N^1(X,\mathbb{R})$*?*

- $\mathrm{Bl}_p\mathbb{P}^2$ shows that the best to hope for is locally Lipschitz in $N^1(X,\mathbb{R})$, and locally $C^{1,1}$ in $\mathcal{B}$

Theorem

$\mathrm{Vol} : N^1(X,\mathbb{R}) \to \mathbb{R}_{\geqslant 0}$ *is locally*

- *Lipschitz on* $N^1(X,\mathbb{R})$ *(Lazarsfeld)*
- C^1 *on* $\mathcal{B}$ *(Lazarsfeld-Mustață, Boucksom-Favre-Jonsson)*
- $C^{1,1}$ *on* $\mathcal{B}$ *(Junyu Cao-T.)*

Expect a wall-chamber decomposition of $\mathcal{B}$ and Vol is smooth in chambers

$D \in \mathcal{E}$ with $\mathrm{Vol}(D) = 0$ exhibit more "patologies". Want to measure this quantitatively

Analytic viewpoint

Natural embedding $c_1 : N^1(X, \mathbb{R}) \to H^{1,1}(X, \mathbb{R})$ induced by $D \mapsto c_1(\mathcal{O}_X(D))$ for D Cartier

Explicitly, fix any smooth hermitian metric h on the line bundle $\mathcal{O}_X(D)$. Its curvature R_h is a closed real $(1,1)$-form given locally by

$$R_h = -i\partial\overline{\partial} \log h$$

Any other metric is of the form $he^{-\varphi}, \varphi \in C^\infty(X, \mathbb{R})$ and

$$R_{he^{-\varphi}} = R_h + i\partial\overline{\partial}\varphi$$

Every smooth representative α of $c_1(\mathcal{O}_X(D))$ is of the form $\alpha = R_h + i\partial\overline{\partial}\varphi$ for some $\varphi \in C^\infty(X, \mathbb{R})$

Examples

Ex. D ample then $c_1(\mathcal{O}_X(D))$ contains a smooth positive definite representative $\alpha > 0$ (i.e. a Kähler metric)

Can take $\alpha = \frac{1}{k}\Phi^*\omega_{\mathrm{FS}}$, where $\Phi : X \overset{|kD|}{\to} \mathbb{P}^N$ embedding, $k \gg 1$

Taking linear combinations, for every $\mathbb{R}$-divisor $D \in \mathcal{A}$ we have $c_1(D)$ contains a Kähler metric, and conversely

Ex. D *semi*ample then $c_1(\mathcal{O}_X(D))$ contains a smooth *semi*positive definite representative $\alpha \geqslant 0$

Again can take $\alpha = \frac{1}{k}\Phi^*\omega_{\mathrm{FS}}$, where $\Phi : X \overset{|kD|}{\to} \mathbb{P}^N$ morphism, k sufficiently divisible

If $c_1(D)$ contains a smooth semipositive representative then $D \in \overline{\mathcal{A}}$ is nef. The converse is false!

Currents

Serre's example: C elliptic curve, E nonsplit extension

$$0 \to \mathcal{O}_C \to E \to \mathcal{O}_C \to 0$$

$X = \mathbb{P}(E)$, $\quad D = \mathcal{O}_{\mathbb{P}(E)}(1)$ is nef but $c_1(D)$ contains no smooth semipositive representative (Yau 74; Demailly-Peternell-Schneider 94)

Demailly: However, given any X and $D \in \mathcal{E}$ pseff, with a smooth representative $\alpha \in c_1(D)$, we can always find $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ quasi-psh such that $T := \alpha + i\partial\overline{\partial}\varphi \geqslant 0$ in the weak sense.

Thus $T \in c_1(D)$ is a closed positive current. Conversely if $c_1(D)$ contains a closed positive current, then $D \in \mathcal{E}$ is pseff.

Lelong numbers

Ex. D Cartier and effective, $s \in H^0(X, \mathcal{O}_X(D))$ nontrivial, and h a smooth Hermitian metric on $\mathcal{O}_X(D)$, then

$$\varphi = \log |s|_h^2$$

is quasi-psh and

$$R_h + i\partial\overline{\partial}\varphi = [D] \geqslant 0$$

$T = \alpha + i\partial\overline{\partial}\varphi \geqslant 0$ closed positive current, $x \in X$, $\alpha \in c_1(D)$. Lelong number

$$\nu(T, x) = \sup\{\gamma \geqslant 0 \mid \varphi(y) \leqslant \gamma \log |x - y| + O(1)\} \in \mathbb{R}_{\geqslant 0}$$

Ex. $D = \sum_i a_i D_i$ effective, then

$$\nu([D], x) = \sum_i a_i \mathrm{mult}_x(D_i)$$

K-trivial manifolds

Question (Motivating question)

Given $D \in \mathcal{E}$ with $\mathrm{Vol}(D) = 0$, what is the least singular closed positive current $T \in c_1(D)$?

Ex. If $K_X \sim \mathcal{O}_X$ and $D \in \overline{\mathcal{A}} \cap \mathcal{B}$ then D is semiample (Kawamata, Hacon-McKernan), hence $c_1(D)$ contains a smooth semipositive representative.

Ex. If $K_X \sim \mathcal{O}_X$ and $D \in \overline{\mathcal{A}}$ Cartier with $\mathrm{Vol}(D) = 0$, then conjecturally $D \equiv D'$ where D' is semiample, hence $c_1(D)$ contains a smooth semipositive representative. Only known for $n = 2$

Theorem (Filip-T.)

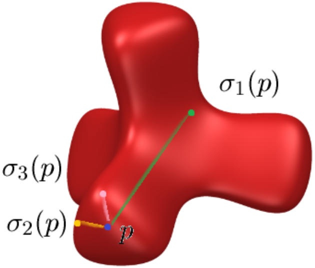
There is X projective K3 with $D \in \overline{\mathcal{A}} \subset N^1(X, \mathbb{R})$ with $\mathrm{Vol}(D) = 0$ such that $c_1(D)$ does NOT contain a smooth semipositive representative.

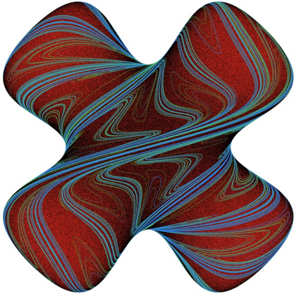
(2,2,2) Examples

$X \subset (\mathbb{P}^1)^3$ generic hypersurface of degree $(2,2,2)$

3 projections to $(\mathbb{P}^1)^2$ exhibit X as a ramified degree 2 cover

$\sigma_1, \sigma_2, \sigma_3 \in \mathrm{Aut}(X)$ covering involutions. $f := \sigma_1 \circ \sigma_2 \circ \sigma_3$ has chaotic behavior under iteration





McMullen

Eigendivisor

$D \in N^1(X, \mathbb{R})$ eigenvector for f^*, $f^* D \equiv \lambda D, \lambda = \log(9 + 4\sqrt{5}) > 1$. We have $D \in \overline{\mathcal{A}}$ and $\mathrm{Vol}(A) = 0$

Cantat: $c_1(D)$ contains a unique closed positive current $T \geqslant 0$. It has zero Lelong numbers.

Theorem (Dynamical rigidity of Kummer; Cantat-Dupont, Filip-T.)

T is smooth if and only if X is Kummer and f induced by an affine map of the corresponding torus.

A generic X is not Kummer since $\dim N^1(X, \mathbb{R}) = 3$

Our proof uses holomorphic dynamics and also Yau's Ricci-flat Kähler metrics on X

K-trivial manifolds

Conjecture (T.)

Suppose $K_X \sim \mathcal{O}_X$, $D \in \overline{\mathcal{A}} \subset N^1(X, \mathbb{R})$, *then* $c_1(D)$ *contains a closed positive current* $T \geqslant 0$ *with* $\nu(T, x) = 0$, *for all* $x \in X$.

True if $D \in \overline{\mathcal{A}} \cap \mathcal{B}$, so we may assume $\mathrm{Vol}(D) = 0$.

False if $K_X \not\sim \mathcal{O}_X$, e.g. in Serre's example D is nef but the only closed positive current in $c_1(D)$ is $[\tilde{C}]$, where $\tilde{C} \subset X$ is the divisor induced by the subbundle $\mathcal{O}_C \to E$

Theorem (Filip-T.)

Conjcture is true if X *projective* $K3$ *with no* (-2)*-curves.*

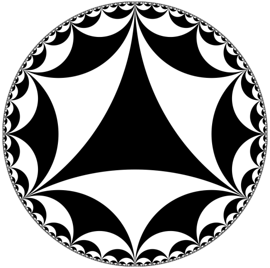
In this case we can even find $T = \alpha + i\partial\overline{\partial}\varphi$ with $\varphi \in C^0(X)$. The current T is unique if $\mathbb{R}c_1(D) \cap H^2(X, \mathbb{Q}) = \{0\}$!

Back to $(2,2,2)$ examples

$X \subset (\mathbb{P}^1)^3$ generic hypersurface of degree $(2,2,2)$

Three projections to $\mathbb{P}^1$, pulling back $\mathcal{O}(1)$ gives three semiample divisor D_1, D_2, D_3 which span $N^1(X,\mathbb{R}) \cong \mathbb{R}^3$

Using $N^1(X,\mathbb{R})$ and its intersection form we obtain a model of $\mathbb{H}^2$ as one sheet of the hyperboloid $\{\mathrm{Vol} = 1\}$

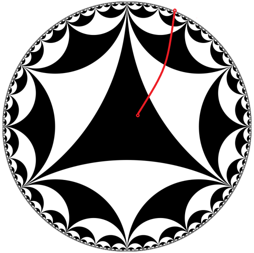


The involutions pullbacks $\sigma_1^*, \sigma_2^*, \sigma_3^*$ generate the symmetries of this tiling

Idea of proof

Theorem (Filip-T.)

Given $D \in \overline{\mathcal{A}}$ *with* $\mathrm{Vol}(D) = 0$, $c_1(D)$ *contains a closed positive current* $T \geqslant 0$ *with* $\nu(T, x) = 0$, *for all* $x \in X$.



Follow a hyperbolic geodesic that ends at $c_1(D)$. In the quotient hyperbolic surface with cusps, it is either divergent (if $c_1(D)$ rational) or recurrent (if $c_1(D)$ irrational). Carefully glue basic estimates.

Higher dimensions

$X \subset (\mathbb{P}^1)^{N+1}$ generic hypersurface of degree $(2, \cdots, 2)$, $N \geqslant 3$

$\sigma_j \in \mathrm{Bir}(X), 1 \leqslant j \leqslant N+1$ birational involutions (not regular!)

These are pseudoautomorphisms, hence can pull back divisors preserving h^0 and Vol

Cantat-Oguiso: X satisfies a strong version of the Kawamata-Morrison cone conjecture: $\overline{\mathcal{A}}$ is rational polyhedral cone, and

$$\mathrm{Bir}(X) \cdot \overline{\mathcal{A}} = \mathcal{M}^e$$

$\mathcal{M}^e$ movable effective cone, whose interior is $\mathcal{B}$

Hyperbolic manifold

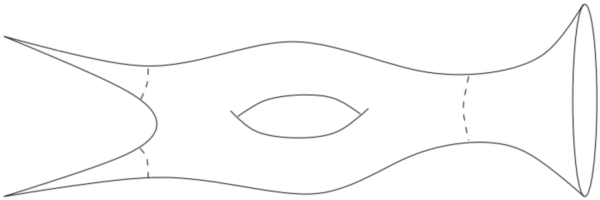
Cantat-Oguiso: There is a $\mathrm{Bir}(X)$-invariant quadratic form Q on $N^1(X,\mathbb{R})$, of signature $(1,N)$. The sheet Δ of the hyperboloid

$$\{v \in N^1(X,\mathbb{R}) \mid Q(v,v) = 1\}$$

which contains some ample divisor is a copy of $\mathbb{H}^N$

$$M^N = (\Delta \cap \mathcal{M}^e)/\mathrm{Bir}(X)$$

infinite-volume hyperbolic real N-manifold, with fundamental domain $\overline{\mathcal{A}} \cap \Delta$ nef classes



Volume function

$\mathrm{Vol} : N^1(X, \mathbb{R}) \to \mathbb{R}$ is $\mathrm{Bir}(X)$-invariant, so descends to a continuous function $\nu : M \to \mathbb{R}_{>0}$, which goes to $+\infty$ in the cusps

Given $D \in \mathcal{E}$ with $\mathrm{Vol}(D) = 0$ and A ample, $\gamma(t) = (D + tA)/\sqrt{Q(D + tA, D + tA)}$ geodesic ray in M

$$\mathrm{Vol}(D + tA) = (Q(D + tA, D + tA))^{\frac{N}{2}} \nu(\gamma(t)) \sim t^{\frac{N}{2}} \nu(\gamma(t))$$

1) When γ is in a compact region in M then $\nu(\gamma(t)) \sim 1$, so $\mathrm{Vol}(D + tA) \sim t^{\frac{N}{2}}$

2) When γ enters a cusp then $\nu(\gamma(t)) \sim t^{1-N/2}$, so $\mathrm{Vol}(D + tA) \sim t$

Conversely, using hyperbolic geometry, we can construct a geodesic ray $\gamma(t)$ in M for which $\nu(t)$ oscillates between $t^{1-\frac{N}{2}}$ and t. Its limiting point at infinity is $D \in \partial\mathcal{E}$ as above. Thus we get:

Pathology of Vol

Theorem (Filip-Lesieutre-T.)

$X \subset (\mathbb{P}^1)^{N+1}$ generic hypersurface of degree $(2, \cdots, 2)$, $N \geqslant 3$. Then there is $D \in \mathcal{E} \subset N^1(X, \mathbb{R})$ with $\mathrm{Vol}(D) = 0$ such that

$$\liminf_{t \downarrow 0} \frac{\log \mathrm{Vol}(D + tA)}{\log t} = 1, \quad \limsup_{t \downarrow 0} \frac{\log \mathrm{Vol}(D + tA)}{\log t} = \frac{N}{2}$$

$$\liminf_{m \to +\infty} \frac{\log h^0(X, \lfloor mD \rfloor + A)}{\log m} = \frac{N}{2}, \quad \limsup_{m \to +\infty} \frac{\log h^0(X, \lfloor mD \rfloor + A)}{\log m} = N - 1.$$

Answers negatively questions of Lesieutre, Nakayama, Lehmann, Eckl, Fujino.

How to obtain sections

On our X, take our $D \in \partial\mathcal{E}$, and A sufficiently ample

Given $m \geqslant 1$ there is a $\phi \in \mathrm{Bir}(X)$ with $\phi^*(\lfloor mD \rfloor + A)$ nef and big.

Theorem

X generic $(2, \ldots, 2)$ hypersurface, L nef and big line bundle, then

$$\mathrm{Vol}(L) \leqslant h^0(X, L) \leqslant C_N \mathrm{Vol}(L)$$

$$h^0(X, \lfloor mD \rfloor + A) = h^0(X, \phi^*(\lfloor mD \rfloor + A)) \approx \mathrm{Vol}(\phi^*(\lfloor mD \rfloor + A))$$
$$= \mathrm{Vol}(\lfloor mD \rfloor + A) \approx \mathrm{Vol}(mD + A) = m^N \mathrm{Vol}\left(D + \frac{1}{m}A\right)$$

and $\mathrm{Vol}\left(D + \frac{1}{m}A\right)$ oscillates between m^{-1} and $m^{-N/2}$