BOUNDEDNESS FOR FIBERED CALABI-YAU VARIETIES

"K-TRIVIAL VARIETIES AND THEIR MODULI"

UCSD, JANUARY 19th 2025

$/\mathbb{C}$

MMP

$$\begin{tikzcd} X \arrow[rr, dashed, "\text{BIRATIONAL}", "\text{(composition of flips + divisorial contractions)}"'] & & X' \arrow[d, "i"] \\ & & Z \end{tikzcd}$$

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3 possibilities for i :

① i is BIRATIONAL, $K_{X'}$ NEF+BIG & $K_Z = i_* K_Z$ IS AMPLE

② i is FIBRATION, $K_{X'}$ NEF & $K_{X'} = i^* H$ AMPLE ON Z

③ i is FIBRATION, $-K_{X'}$ AMPLE ALONG THE FIBERS OF i

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3 IMPORTANT BUILDING BLOCKS FOR CLASSIFICATION

1. LOG CANONICAL MODELS

$$K_X + \Delta > 0$$

2. LOG K-TRIVIAL PAIRS

$$K_X + \Delta \equiv 0$$

3. LOG FANO PAIRS

$$K_X + \Delta < 0$$

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2nd part of classification of algebraic varieties:

CONSTRUCT MODULI SPACES

NAIVE 3-STEP RECIPE TO BUILD A MODULI SPACE

1. BOUNDEDNESS: IDEALLY, WE WANT A PROPER & SEPARATED MODULI SPACE; HENCE, WE SHOULD CHECK THAT THE LOG PAIRS PARAMETRIZED BELONG TO JUST FINITELY MANY FAMILIES OF DEFORMATIONS.

   THIS WILL REQUIRE FIXING INVARIANT: $\mathcal{M}_g$

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$$\mathcal{M}_g \subset \overline{\mathcal{M}}_g = \{\text{STABLE GENUS } g \text{ CURVES}\}$$

$$\text{STABLE} = \text{NODAL} + \omega_C \text{ AMPLE}$$

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③ CONSTRUCT THE MODULI SPACE : CHOOSE YOUR POISON :
GIT, VGIT, KSBA, K-MODULI, ...

DEFINITION [BOUNDEDNESS]

Let $\mathcal{C} = \{ X_i \}_{i \in I}$ be a collection of proper algebraic varieties.

We say that $\mathcal{C}$ is **BOUNDED** if

$$\exists \ \mathcal{X} \xrightarrow{\text{proper morphism}} T \quad \text{such that} \quad \forall i \in I, \ \exists t_i \in T$$

$$\text{s.t. } \mathcal{X}_{t_i} \simeq X_i$$

ISOMORPHISM

FINITE TYPE

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$$

FINITE TYPE

BIRATIONAL ISOMORPHISM

Let $\mathcal{C}$ be a collection of varieties $\{X_i\}_{i \in I}$ all of the same dimension $d \in \mathbb{Z}_{>0}$. Then:

① $\mathcal{C}$ is BOUNDED $\Longleftrightarrow$ $\exists V \in \mathbb{Z}_{>0}$ s.t. $\forall i \in I$,

$\exists H_i$ VERY AMPLE on X_i

<u>and</u> $H_i^d \leq V$ ← indep. of (X_i)

② $\mathcal{C}$ is BIRATIONALLY BOUNDED $\Longleftrightarrow$ $\exists V' \in \mathbb{Z}_{>0}$ s.t. $\forall i \in I$,

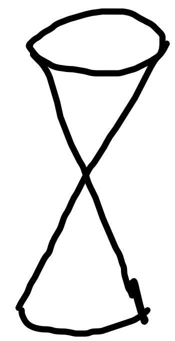
$\exists H_i$ BIG on X_i

<u>and</u> $\varphi_{|H_i|} : X_i \dashrightarrow \mathbb{P}(H^0(H_i)^\vee)$

is birational w/ degree of the image $\leq V'$.

EXAMPLE

deg $= n$ RAT'L NORMAL CURVE

$\mathcal{C} = \Big\{ X_n = \mathbb{P}(1,1,n) =$  $\Big\}_{n \in \mathbb{N}}$

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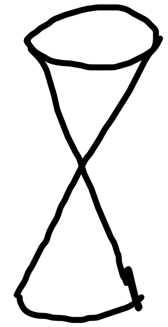
$\mathcal{C} = \Big\{ \; X_n = \mathbb{P}(1,1,n) = $  $\Big\}$

$\mathcal{C}$ is BIRATIONALLY BOUNDED.

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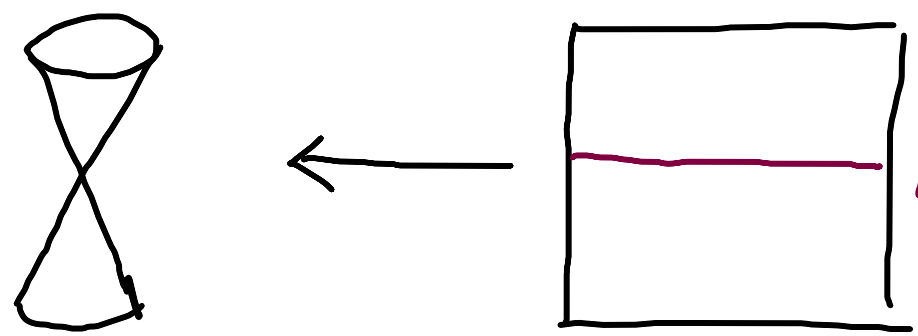
$\mathcal{C}$ IS NOT BOUNDED :



$X_n \qquad \mathbb{F}_n \qquad E_n^2 = -n$

$$a(E_n, K_{X_n}) = -1 + \frac{2}{n} \xrightarrow{n \to \infty} -1$$

DEFINITION [LOG BOUNDEDNESS]

Let $\mathcal{C}=\{(X_i,\Delta_i)\}_{i\in I}$ be a collection of proper log pairs.

We say that $\mathcal{C}$ is **LOG BOUNDED** if

$\exists\,(\mathcal{X},\mathcal{D})$ ← reduced divisor, such that $\forall i\in I$, $\exists\, t_i\in T$

$$\begin{array}{c}\mathcal{X}\\ \Big\downarrow {\scriptstyle\text{proper morphism}}\\ T\end{array}$$

T ← FINITE TYPE

s.t. $h_i:\mathcal{X}_{t_i}\xrightarrow[\text{ISOMORPHISM}]{} X_i$

$$h_i(\operatorname{supp}(\mathcal{D}_{t_i}))=\operatorname{supp}(\Delta_i)$$

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WHY IS BOUNDEDNESS AN INTERESTING PROPERTY?

IF A COLLECTION $\mathfrak{D}$ OF VARIETIES IS BOUNDED THEN WE CAN CONCLUDE THAT MANY INVARIANTS OF VAR'S $\in \mathfrak{D}$ CAN ONLY ATTAIN FINITELY MANY VALUES.

THEOREM [VERDIER] LET $h : \mathcal{X} \longrightarrow T$ BE A PROPER MORPHISM OF FINITE TYPE VARIETIES.

$\exists\, U \subseteq T$ ZARISKI OPEN $(\neq \emptyset)$ SUCH $\mathcal{X}|_U \xrightarrow{h|_U} U$ IS A TOPOLOGICALLY TRIVIAL FIBRATION (IN THE EUCLIDEAN TOP.)

BOUNDESS FOR LOG CANONICAL MODELS : [HACON - McKERNAN - XU]

FIX $d \in \mathbb{Z}_{>0}$, $v \in \mathbb{R}_{>0}$, $I \subset [0,1]$ DCC SET.

$$\left\{ (X, \Delta) \;\middle|\; \begin{array}{ll} (X, \Delta) \text{ SLC} & K_X + \Delta \text{ ample} \\ \dim X = d & (K_X + \Delta)^d = v \end{array} \right\}$$

is LOG BOUNDED

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BOUNDESS FOR LOG FANO PAIRS : [BIRKAR]

FIX $d \in \mathbb{Z}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$.

$$\left\{ X \;\middle|\; \begin{array}{l} \exists\, (X,\Delta) \ \varepsilon\text{-klt} \quad -(K_X+\Delta) \text{ ample} \\ \dim X = d \end{array} \right\}$$

is BOUNDED

BOUNDEDNESS (OR LACK THEREOF) IN THE K-TRIVIAL WORLD

QUESTION What happens for K-trivial varieties / log pairs?

- dim = 1 , Elliptic curves are BOUNDED : $\mathbb{A}^1_j$

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- dim = 1, Elliptic curves are BOUNDED : $\mathbb{A}^1_j$;
- dim = 2, projective Abelian / K3 surfaces are unbounded : $\mathcal{F}^{2d}$, $\mathcal{A}_{2,(d_1,d_2)}$;

  but they are topologically bounded (via non-algebraic Kähler models)

  Enriques & bi-elliptic surfaces are bounded

QUESTION What happens for K-trivial varieties / log pairs?

THEOREM [BEAUVILLE, BOGOMOLOV]

Let X be a smooth compact Kähler manifold w/ $K_X \equiv 0$.

$$\exists \quad X \xleftarrow{\text{étale}} X' := T \times \prod Y_i \times \prod Z_j$$

T: $\mathbb{C}$-torus

Y_i: ICY
simply connected
$$h^{i,0}(X) = \begin{cases} 0 & 0 < i < \dim X \\ 1 & \text{otherwise} \end{cases}$$

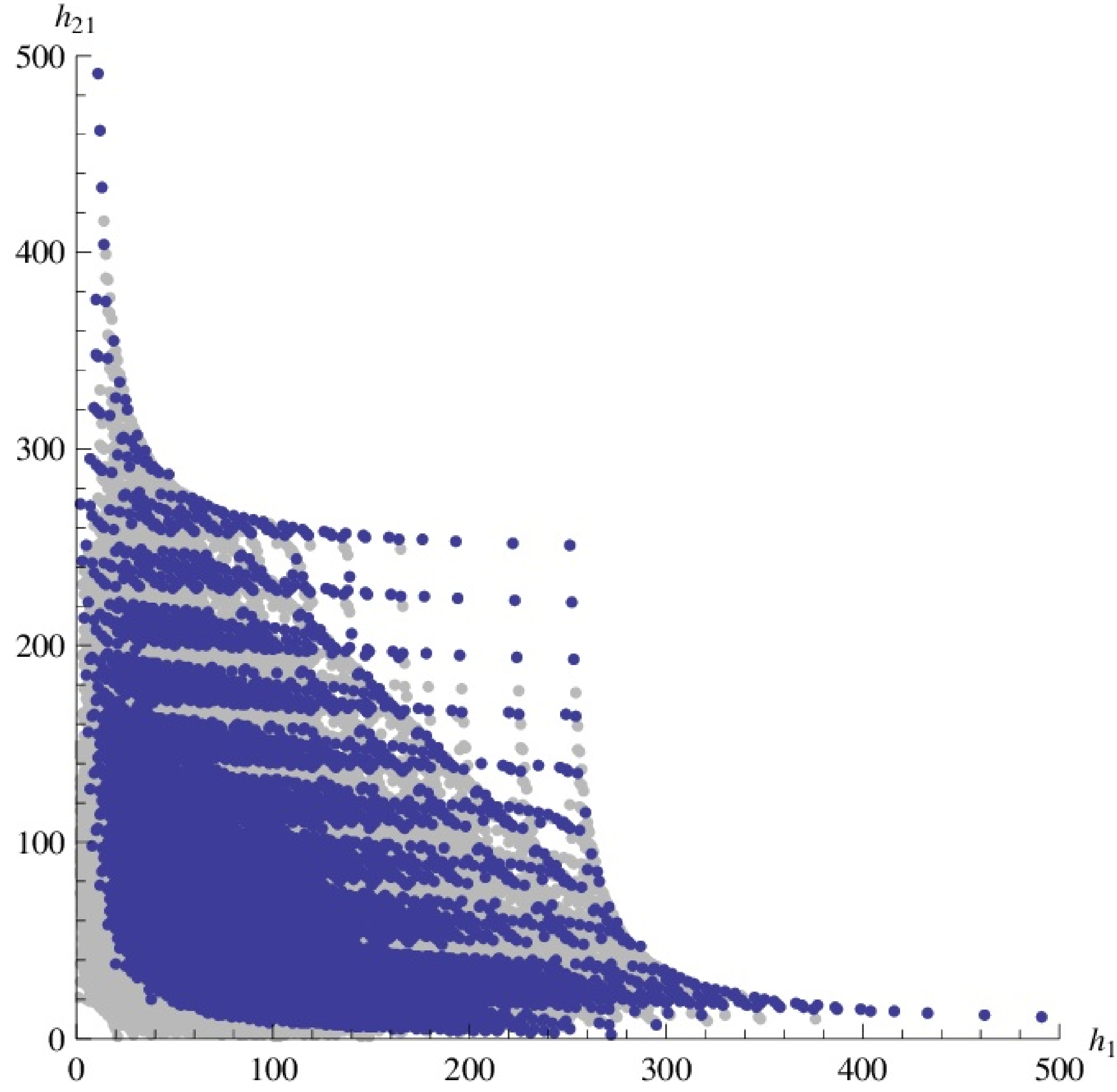
Z_j: IHS
simply connected
+ hol. symplectic form.

BOUNDEDNESS (OR LACK THEREOF) IN THE K-TRIVIAL WORLD

- dim $= 1$, Elliptic curves are BOUNDED : $\mathbb{A}^1_j$
- dim $= 2$, projective Abelian/K3 surfaces are unbounded : $\mathcal{F}^{2d}$, $\mathcal{A}_{2,(d_1,d_2)}$; Enriques & bi-elliptic surfaces are bounded
- dim > 2, Abelian/HK varieties are alg. unbounded; what about CY varieties?

CONJECTURE / FANTASY : The set of CY 3-folds is bounded topologically.

Yau — Reid



h_{21}
500
400
300
200
100
0
0
100
200
300
400
500
h_{11}

FINALLY SOME GOOD NEWS

Elliptic fibration : $f : X \longrightarrow Y$ fibration of relative dim $= 1$ w/ elliptic general fiber.

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THEOREM [GROSS, '94] The collection

$$\left\{ X \;\middle|\; \begin{array}{l} (X, 0) \text{ TERMINAL 3-FOLD}, \; K_X \sim 0, \; h^1(\mathcal{O}_X) = 0 = h^2(\mathcal{O}_X), \\ \exists\, f : X \longrightarrow Y \text{ elliptic w/ } Y \text{ rational} \end{array} \right\}$$

is BIRATIONALLY BOUNDED (up to flops).

$$\begin{tikzcd} \mathcal{H} \arrow[d] & \supset & \mathcal{H}_{\bar{t}} \arrow[d] \arrow[r, dashed, "\text{birat'l}"] & X \\ T & \ni & \bar{t} & \end{tikzcd}$$

isomorphism in codim 1

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COROLLARY: THERE ARE ONLY FINITELY MANY TOPOLOGICAL TYPES OF ELLIPTIC CY 3-FOLDS.

$\exists\, C_{p,q} \in \mathbb{Z}_{>0}$ S.T. $|h^{p,q}(X)| \leq C_{p,q}$.

Idea of proof [GROSS]

① $f : X \longrightarrow Y$ elliptic CY $\xrightarrow{\text{Jac}}$ $f_J : X_J \longrightarrow Y$

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③ Undo the correspondence Jac :

Q. How many elliptic CYs correspond to the same $X_J \xrightarrow{f_J} Y$?

[GROSS-DOLGACHEV] The correspondence is finite, and upper-semicontinuous in families (via $Ш_Y(X_J)$)

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$$\Longrightarrow \{ X \mid (X,0) \text{ TERMINAL 3-FOLD}, K_X \sim 0, h^1(\mathcal{O}_X) = 0 = h^2(\mathcal{O}_X), \exists f : X \longrightarrow Y \text{ elliptic w/ } Y \text{ rational} \}$$

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THEOREM [F-H-S] Let
$$\begin{tikzcd} \mathcal{X} \arrow[d] \arrow[r] & \mathcal{Y} \arrow[dl] \\ T & \end{tikzcd}$$
be a family of elliptic CY 3-folds.

$\Longrightarrow$ $\exists$ finitely many
$$\begin{tikzcd} \mathcal{X}^i \arrow[d] \arrow[r] & \mathcal{Y} \arrow[dl] \\ T & \end{tikzcd}$$
s.t. $\forall\ \mathcal{X}_t$ all models isomorphic in codim. 1 over $\mathcal{Y}_t$ appear as fibers of one of the $\mathcal{X}^i$

FINALLY SOME GOOD NEWS

Elliptic fibration: $f : X \longrightarrow Y$ fibration of relative dim $= 1$ w/ elliptic general fiber.

THEOREM [Di Cerbo - S ; Birkar - Di Cerbo - S]

Fix $d \in \mathbb{Z}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$. The collection

$$\mathfrak{C}^{CY,ell}_{d,\varepsilon} = \left\{ X \;\middle|\; \begin{array}{l} (X,0) \text{ is } \varepsilon\text{-klt},\ K_X \sim_{\mathbb{Q}} 0,\ \exists f : X \xrightarrow[\text{elliptic}]{} Y \\ \text{s.t. } Y \text{ is RC},\ \exists \sigma : Y \dashrightarrow X \text{ rat'l section}, \\ h^i(\mathcal{O}_X) = 0 \quad \forall\, 0 < i < \dim X \end{array} \right\}$$

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COROLLARY : $\exists\, C_{p,q} \in \mathbb{Z}_{>0}$ s.t. $|h^{p,q}(X)| \leq C_{p,q}$ $\forall X \in \mathcal{C}^{CY, ell}_{d, \varepsilon}$.

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[FILIPAZZI, JIAO] : SUFFICES TO ASSUME EXISTENCE OF A SECTION OF degree $\leq \overline{C}$ (independent of X)

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2. study boundedness of the variation for the $\{ f_i \}_{i \in I}$:

- $f_i : X_i \longrightarrow Y_i \quad \rightsquigarrow \quad Y_i \overset{\mu_i}{\dashrightarrow} \mathcal{M}_{\text{Fibers of } f_i}$

$$y \longmapsto [f_i^{-1}(y_i)]$$

GOAL: use CY condition ($K \equiv 0$ + $h^{1,0} = 0$) to show "boundedness" of the μ_i

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ALTERNATIVELY: TRY TO CONSTRUCT A HORIZONTAL POLARIZATION D_i on X_i f_i-ample w/ bounded volume on the fibers [Jiao]

BASES OF FIBERED CY

$$f : X \longrightarrow Y \qquad K_X \equiv 0$$

f fibration

CANONICAL BUNDLE FORMULA :

$$K_X \sim_{\mathbb{Q}} f^*(K_Y + B_Y + M_Y)$$

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CANONICAL BUNDLE FORMULA :

$$K_X \sim_{\mathbb{Q}} f^*(K_Y + B_Y + M_Y)$$

≥ 0 divisor

$\mathbb{Q}$ - linear equivalence class of divisors

$$B_Y := \sum \left(1 - \mathrm{lct}_{\eta_D}\left(X; f^{-1}(D)\right)\right) D$$

D prime div $\subset Y$

$$\begin{tikzcd}
 & Y^{res} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
Y \arrow[rr, dashed] & & \mathcal{M}_{Fibers}
\end{tikzcd}$$

$M_Y = p_* q^* \mathcal{H}_{Fibers}$ ⟵ big + nef

CONJECTURE [Shokurov, McKernan-Prokhorov]

Fix $d \in \mathbb{Z}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$. The collection

$$\mathcal{C}^{RC,lcy}_{d,\varepsilon} = \left\{ X \;\middle|\; \begin{array}{l} \dim X = d, \quad X \text{ is } RC, \ \exists\, \Gamma \geq 0 \\ \text{s.t. } (X, \Gamma) \text{ is } \varepsilon\text{-klt}, \ K_X + \Gamma \equiv 0 \end{array} \right\}$$

is BOUNDED.

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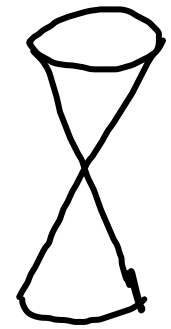
$\mathcal{C} = \{ X_n = \mathbb{P}(1,1,n) =$  $\}$

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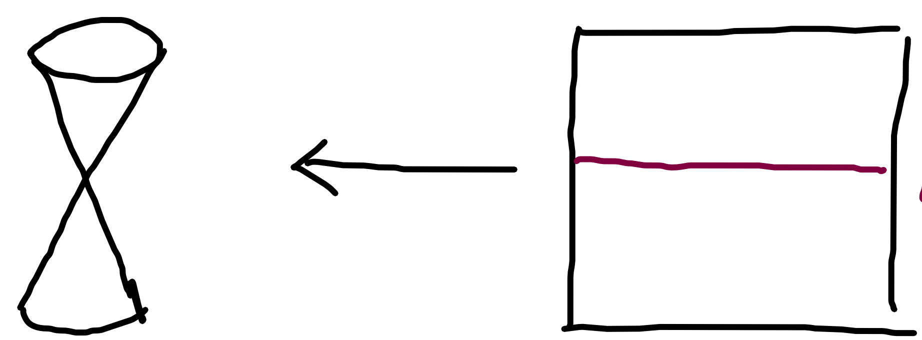

X_n $\qquad$ $\mathbb{F}_n$

$E_n^2 = -n$

$$a(E_n, K_{X_n}) = -\frac{2}{n} \xrightarrow{n \to \infty} -1$$

CONJECTURE [Shokurov, McKernan-Prokhorov]

Fix $d \in \mathbb{Z}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$. The collection

$$\mathcal{C}^{RC, lcy}_{d,\varepsilon} = \left\{ X \;\middle|\; \begin{array}{l} \dim X = d, \ X \text{ is } RC, \ \exists \Gamma \geq 0 \\ \text{s.t. } (X, \Gamma) \text{ is } \varepsilon\text{-Klt}, \ K_X + \Gamma \equiv 0 \end{array} \right\}$$

is BOUNDED.

EXAMPLE We have already seen that bounding the sing's is necessary. Also RCness is necessary:

$$\left(\mathbb{P}^1 \times K3, \ \tfrac{1}{2} p_1^{*} (\{0\} + \{1\} + \{\lambda\} + \{\infty\})\right) \text{ is } \tfrac{1}{2}\text{-klt}$$

is log CY but certainly is NOT BOUNDED.

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is BOUNDED.

THEOREM [Alexeev, $d=2$; B-DC-S $d=3$; Birkar $d>3$]

Fix $d \in \mathbb{Z}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$. The collection $\mathcal{C}^{RC,lcy}_{d,\varepsilon}$ is BIRATIONALLY BOUNDED (up to flops).

STRATEGY OF PROOF

$$\begin{tikzcd} (X,\Gamma) \arrow[r, dashed, "K_X\text{-MMP}"] & (X',\Gamma') \arrow[d, "\text{MFS}"] \\ & Z \end{tikzcd}$$

$\Gamma \neq 0$

if X is RC
$\Longrightarrow$ Z is RC

CBF : $\exists\ \Theta \geq 0$ s.t.
(Z, Θ) is klt CY

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$\Gamma \not\equiv 0$

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CBF : $\exists\ \Theta \geq 0$ s.t.

(Z,Θ) is klt CY

NEW THM [BIRKAR '23]

If (X,Γ) is ε-klt

$\Longrightarrow \exists\ \delta = \delta(\varepsilon, \dim X)$

s.t. (Z,Θ) is δ-klt.

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$\Uparrow$

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FINALLY SOME GOOD NEWS

Elliptic fibration : $f : X \longrightarrow Y$ fibration of relative dim $= 1$ w/ elliptic general fiber.

THEOREM [Di Cerbo - S ; Birkar - Di Cerbo - S]

Fix $d \in \mathbb{Z}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$. The collection

$$\mathcal{C}^{CY,ell}_{d,\varepsilon} = \left\{ X \;\middle|\; \begin{array}{l} (X,0) \text{ is } \varepsilon\text{-klt},\ K_X \sim_{\mathbb{Q}} 0,\ \exists\, f : X \xrightarrow[\text{elliptic}]{} Y \\ \text{s.t. } Y \text{ is RC},\ \exists\, \sigma : Y \dashrightarrow X \text{ rat'l section}, \\ h^i(\mathcal{O}_X) = 0 \quad \forall\, 0 < i < \dim X \end{array} \right\}$$

is BIRATIONALLY BOUNDED (up to flops).

SKETCH OF PROOF

$$\begin{tikzcd} X \arrow[r, "f"'] & Y \arrow[l, dashed, bend left, "G"] \end{tikzcd}$$

$f: X \longrightarrow Y$ elliptic, K-trivial

X ε-klt

Y RC

① Using CBF: $K_Y + B_Y + M_Y \equiv 0$

$\Longrightarrow$ by Kodaira + effective semiampleness of $\mathbb{M}_Y$

$\exists\ \Theta_f$ on Y s.t. (Y, Θ_f) is $\underset{\eta(d)}{\underset{\parallel}{\eta}}$-klt + log CY

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$f : X \to Y$ elliptic $\quad$ K-trivial

$X \quad \varepsilon$-klt

$Y \quad$ RC

(1) Using CBF : $K_Y + B_Y + M_Y \equiv 0$

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$\exists\ \Theta_f$ on Y s.t. (Y, Θ_f) is $\underset{\eta(d)}{\overset{\|}{\eta}}$-klt + log CY

[BirKar]: The set of bases $\{ Y \mid$ RC bases of elliptic CYs $\}$ is BIRATIONALLY BOUNDED (up to flops).

WCA $\exists\ H$ very ample on Y s.t. $H^{d-1} \le B$ (indep. of Y)

SKETCH OF PROOF

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl, dashed, no head] \\ & G \arrow[ul, dashed] & \end{tikzcd}$$

elliptic — K-trivial

X — ε-klt

Y — RC

WCA $\exists$ H very ample on Y s.t. $H^{d-1} \leq B$ (indep. of Y)

② Let $\Sigma = \overline{G(Y)}$. Running a $(K_X + t\Sigma)$-MMP $(0 < t \ll 1)$

WCA that Σ is f-ample.

SKETCH OF PROOF

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[l, dashed, bend left, "G"] \end{tikzcd}$$

f elliptic   K-trivial

X ε-klt

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[Di Cerbo - S] : in these hypotheses (X, Σ) is plt + $\mathrm{Diff}_\Sigma(0) = 0$

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[Kawamata's length of ERs] : $K_X + \frac{1}{2}\Sigma + (2d+2) f^* H$ is ample

$+$ $\mathrm{Vol}(t\Sigma + H)$ has derivative controlled by B.

SKETCH OF PROOF

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elliptic		K - trivial
	X	ε - klt
	Y	RC

WCA $\exists$ H very ample on Y s.t. $H^{d-1} \leq B$ (indep. of Y)

[Di Cerbo - S] : (X, Σ) is plt $\Longrightarrow$ $\left(X, \frac{1}{2}\Sigma + (2d+2)f^*H\right)$ $\frac{\varepsilon}{2}$-klt

[Kawamata's length of ERs] : $K_X + \frac{1}{2}\Sigma + (2d+2)f^*H$ is ample

$$\mathrm{Vol}\left(K_X + \tfrac{1}{2}\Sigma + (2d+2)f^*H\right)$$
$$\parallel$$
$$\mathrm{Vol}\left(\tfrac{1}{2}\Sigma + (2d+2)f^*H\right) \leq \varphi(B)$$

[HMX] : ① $\left\{\left(X, \frac{1}{2}\Sigma + (2d+2)f^*H\right)\right\}$ is LOG BIR. BOUNDED.

② Since sing's + coeff's are bounded, it is actually LOGBOUNDED.

EXTENDING TO HIGHER DIMENSION

QUESTION Fix $d \in \mathbb{Z}_{>0}$.

Let $\mathcal{C}^{\text{fibered}}_{d,CY} := \left\{ X \;\middle|\; \begin{array}{l} \dim X = d,\ K_X \sim 0,\ (X,0) \text{ canonical} \\ X \text{ is CY},\quad \exists\, f : X \to Y \text{ fibration} \end{array} \right\}$

Is $\mathcal{C}^{\text{fibered}}_{d,CY}$ bounded?

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SOME CHALLENGING QUESTIONS TO ANSWER:

- Why should the fibers of $X \in \mathcal{C}^{\text{fibered}}_{d,CY}$ be at all bounded?

WORKING HYPOTHESIS: yes, if we assume some topological boundedness for the fibers.

DIVIDE & CONQUER APPROACH

Let $\{f_i : X_i \longrightarrow Y_i\}_{i \in I}$ be a collection of fibered varieties.

To study its boundedness we can try to split the task into 2 parts:

1. Study boundedness for the set of bases $\{Y_i\}_{i \in I}$;

2. study boundedness of the variation for the $\{f_i\}_{i \in I}$:

$$- \quad f_i : X_i \longrightarrow Y_i \quad \rightsquigarrow \quad Y_i \overset{\mu_i}{\dashrightarrow} \mathcal{M}_{\substack{\text{Fibers} \\ \text{of } f_i}}$$

$$y \longmapsto [f_i^{-1}(y_i)]$$

GOAL: use CY condition $(K \equiv 0 + h^{1,0} = 0)$ to show "boundedness" of the μ_i

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- Are the bases still bounded? The answer should still be YES under the WORKING HYPOTHESIS above [Fujino - Mori, 00]