Rational Chow ring of the universal moduli space of semistable rank two bundles over genus two curves

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K-trivial varieties and their moduli
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The $\mathbb{G}_m$-rigidification

- There are two versions of the universal stack of slope (semi)stable bundles over curves in existing literature.
- The objects of both stacks $\mathfrak{U}(r, d, g), \mathcal{U}(r, d, g)$ consist of families of smooth curves of genus g with a bundle of rank r and degree d with morphisms in $\mathcal{U}(r, d, g)$ defined upto twisting by a line bundle.

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- The natural map $\mathfrak{U}(r,d,g) \to \mathcal{U}(r,d,g)$ is a $\mathbb{G}_m$-rigidification.
- These stacks, in particular their geometry, has been studied in numerous papers including Schmitt (2005), Melo & Viviani (2010), Fringuelli (2018), Grimes (2020), Larson (2024) etc.

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Assuming the ground field to be of characteristic 0 and algebraically closed, we calculate $A^*(\mathcal{U}(2, d, 2))$ for all degrees d.

First Steps & Assumptions

We work over an algebraically closed ground field of characteristic 0.

- We calculate $A^*(\mathcal{U}(2,d,2))$ by calculating the Chow ring of its coarse moduli space $U(2,d,2)$.
- We start with an étale S_6-cover of M_2, call it M_*.
- More concretely, $M_* \subset \mathbb{A}^3$ is the coarse moduli space for the stack $\mathcal{H}^w_{2,6}$ as considered by Edidin& Hu (2022).
- We construct a "universal" family of genus 2 curves $\mathscr{C}_* \xrightarrow{\pi} M_*$ with sections $\{\sigma_i\}_{1\leq i\leq 6}$ parametrizing the Weierstrass points on fibers of π.

S_6-actions

- The map $M_* \to M_2$ defined by π, is the S_6-quotient map on $M_* \simeq M_{0,6}$.
- This S_6-action lifts to $\mathscr{C}_*$ via π, permuting the sections $\{\sigma_i\}_{1\leq i\leq 6}$ correspondingly.
- We use the S_6-equivariance and the hyperelliptic involution of π to calculate $A^*(U(2,d,2))$ via $\tilde{U}(2,d,2)$.
- We start with universal Jacobians for the family $\mathscr{C}_* \to M_*$ and then move to even and odd degree cases for rank 2.

"Universal" Jacobians

- $J_*(d)$ be the relative Jacobian defined using π.
- The S_6-equivariance of π defines an S_6-action on $J_*(d)$.
- The quotient $J_*(d)/S_6$ is generically a double cover of $J_{d,2}$ due to the involution on $\mathscr{C}_*$.

$$\begin{tikzcd}
J_*(d) \arrow[rr, "/S_6"] \arrow[ddr] & & J_*(d)/S_6 \arrow[rr, "q_h"] & & J_{d,2} \arrow[ddl] \\
& \mathscr{C}_* \arrow[r, "/S_6"] \arrow[d] & \mathscr{C}_*/S_6 \arrow[r, "q_h"] & M_{2,1} \arrow[d] & \\
& M_* \arrow[rr] & & M_2 &
\end{tikzcd}$$

$A^*(J_*(d))$

Analysis of the S_6-action and the involution on $J_*(d)$ shows that $A^*(J_{d,2}) \simeq A^*(J_*(d))$. Giving us the first Theorem, which provides an alternate proof to Theorem 1.1 in Larson (2024) for $J_{d,2}$.

Theorem

The Chow ring of $J_*(d)$ is given by

$$A^*(J_*(d)) \simeq \mathbb{Q}[\Theta_d]/(\Theta_d^3)$$

for all $d \in \mathbb{Z}$. Consequently, the Chow rings of $J_{2,d}$ are given by

$$J_{2,1} \simeq \mathbb{Q}[\Theta]/(\Theta^3), J_{2,2} \simeq \mathbb{Q}[Z]/(Z^3)$$

Existing techniques

- We have that $|2\Theta_C| \xrightarrow[\sim]{D_C} SU_C(2)$ for any curve C of genus 2 by Narasimhan & Ramanan (1969).
- The map D_C is used to construct $U_C(2,0)$ from $SU_C(2)$ by considering the map

$$SU_C(2) \times J^0(C) \to U_C(2,0)$$

making $U_C(2,0)$ the quotient of the product by 2-torsion points in $J^0(C)$.

Why M_*?

- Extending arguments in [4] directly over M_2 is difficult since the diagonal action of Γ_2 on $|2\Theta_C| \times J^0(C)$ is now an étale equivalence relation.

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- To work around this, we move the entire discussion over to M_* from M_2 and observe that Γ_* splits. Making this étale equivalence relation a finite group action.
- Using the section σ_1, we define a Poincaré bundle on $\mathscr{C}_J$ and use it to construct the universal extension space as described in [5].

These are all the ingredients needed to extend results from [4] to the universal setting.

Quotients and the Chow groups

$$|2\Theta_*|_\pi \times_{M_*} J_*(d) \xrightarrow{q_\Gamma} \tilde{U}(2,2d,2) \xrightarrow{q_{S_6}} U_*(2,2d,2) \xrightarrow{q_h} U(2,2d,2)$$

The sequence of quotient maps gives the following theorem which coincides with Fringuelli (2018) in codimension 1.

Theorem

Let $\nu \in A^1(\tilde{U}(2,2d,2))$ be the divisor which pulls back to $\pi_\Theta^*\zeta$ and $\tilde{Z}_d := \det^* Z_{2d}$. We have

$$A^*(\tilde{U}(2,2d,2)) \simeq \mathbb{Q}[\nu, \tilde{Z}_d]/(\nu^4, \tilde{Z}_d^3)$$

for all $d \in \mathbb{Z}$ where $\tilde{Z}_d = \det_d^* Z_d$. Furthermore, $\nu + \frac{1}{2}\tilde{Z}_d$ can be identified with the generalized Θ-divisor if d is odd.

Geometry of $U_C(2, L)$

We use Bertram (1992) and apply it to case $d = 3, g = 2$. This provides a recipe to construct $U_C(2, L)$ from the extension space $\mathbb{P}_L := \mathbb{P}(H^1(C, L^\vee))$ via a single blowup-blowdown pair.

$$\begin{tikzcd}
 & E_L \arrow[r, hook] \arrow[d] & bl_C(\mathbb{P}_L) \arrow[r, "\Phi_L"] \arrow[d, "\sigma_L"] & U_C(2, L) \\
C \arrow[r, hook] & Q_L \arrow[r, hook] & \mathbb{P}_L \arrow[ur, dashed, "\phi_L"] &
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\end{tikzcd}$$

- E_L maps onto the unique quadric hypersurface containing $C \hookrightarrow Q_L$.
- Q_L is a quadric cone iff L has a basepoint.

Generalizing over M_*

- Due to the lack of a canonical section $M_2 \to J_{3,2}$, there is no direct way to go from fixed determinant to the entire Jacobian.
- We consider a universal version of the previous diagram by using the Poincaré bundle ξ on $\mathscr{C}_J := \mathscr{C}_* \times_{M_*} J_*(3)$.

$$
\begin{tikzcd}
\xi \arrow[d] & E_\xi \arrow[r, hook] \arrow[d] & bl_{\mathscr{C}_J}(\mathbb{P}_\xi) \arrow[r, "\Phi_\xi"] \arrow[d, "\sigma_\pi"] & \tilde{U}(2,3,2) \\
\mathscr{C}_J \arrow[r, hook] & Q_\xi \arrow[r, hook] & \mathbb{P}_\xi &
\end{tikzcd}
$$

We still have that σ_π, Φ_ξ are a blowup-blowdown pair and move through this diagram to calculate $A^*(\tilde{U}(2,3,2))$.

Geometry of E_ξ, Φ_E

- Q_ξ is a family of smooth quadric surfaces degenerating into quadric cones over $J_*(3)$. Making E_ξ a family of $\mathbb{F}_0$'s degenerating to $\mathbb{F}_2$.
- $B_* := \Phi_\xi(E_\xi)$ is a $\mathbb{P}^1$-fibration over $J_*(3)$. The map Φ_E giving the Hirzebruch surface vibration.
- It can also be seen that B_* is the universal Brill-Noether locus of bundles in $\tilde{U}(2,3,2)$ with 2 sections.

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- Excision on $J_*(3)$ helps with stratifying E_ξ. We use sections $\{\sigma_i\}_{1\leq i\leq 6}$ to construct various loci in E_ξ.

Final Theorem

Calculating the Chow rings of $\mathscr{C}_J, \mathbb{P}_\xi, \tilde{\mathbb{P}}_\xi$ and analyzing the blowdown Φ_ξ gives the following theorem which coincides with Fringuelli (2018) in codimension 1.

Theorem

$$A^*(\tilde{U}(2,3,2)) = \mathbb{Q}[H_U, \Theta_U, B_*]/(\Theta_U^3, H_U^2\Theta_U^2 - 4B_*\Theta_U^2, B_*H_U^2 - B_*H_U\Theta_U + \frac{1}{2}B_*\Theta_U^2, B_*H_U^2 - B_*^2, H_U^3 + H_U^2\Theta_U + \frac{1}{2}H_U\Theta_U^2 - 4B_*H_U)$$

Therefore, the Chow ring of $U(2,3,2)$ is isomorphic to the above via $q_{S_6}^* \circ q_h^*$.

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