Canonical bundle formula and equivalence between non-vanishing and Campana–Peternell conjectures

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Conjectures

Non-vanishing conjecture

X smooth projective.

$$K_X \text{ pseudo-effective } \implies K_X \text{ effective.}$$

Campana–Peternell (CP) conjecture

X smooth projective, D effective $\mathbf{Q}$-divisor on X.

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- Question: Does non-vanishing imply the Campana–Peternell conjecture?
- Today: See how the canonical bundle formula comes in and what it suggests!

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- Actually, the prediction is that $mK_X - f^*H$ is effective for some $m > 0$. If this is the case, then $\kappa(X) = \kappa(F) + \dim Y$ by Mori, and $\kappa(F) \geq 0$ since K_F is peff (Non-vanishing!).

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- Using more reductions, it is enough to assume $\kappa(F) = 0$.

Schnell's theorem

Theorem (Schnell)

Let $f\colon X \to Y$ be an algebraic fiber space between smooth projective varieties. Assume $\kappa(F) \geq 0$ and suppose that $L_0 = m_0 K_X - f^*H$ peff for some $m_0 > 0$, and **assume that** K_Y **is peff**. Then $mK_X - f^*H$ is effective for some $m > 0$.

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  with better regularity, as $m_1 \to \infty$.
- However, we need to assume that K_Y is peff, due to the relative pluri-canoincal bundle $K_{X/Y}$.

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- This metric is by definition, semi-positive. The length $I(v)$ becomes finite as $m \to \infty$.
- If $h^0(X, mK_X + L) = 1$, then the inf goes away, and the curvature of h_{BK} is just $[v = 0]$, where $v \in H^0(X, mK_X + L)$.

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- However, the metric tends to be singular near the singular fibres.
- Heuristically, a morphism to negative K_Y has a lot of singular fibers (hyperbolicity).
- Can we examine the contribution of the singular fibers to overcome the non-pseudo-effectivity of K_Y?

Canonical bundle formula

- $f : X \to Y$ with $\kappa(F) = 0$, and assume $m_0 K_X - f^* H$ pseudo-effective.
- Write $K_X \sim_{\mathbf{Q}} f^*(K_Y + B_Y + M_Y) + \Delta$, where

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- I will present two possible strategies of dealing with this Δ

Main result

Theorem

Let $f\colon X \to Y$ be algebraic fiber spaces between smooth projective varieties,
$\kappa(F) = 0$ and $L_0 = m_0 K_X - f^*H$ is peff for some $m_0 > 0$. Write

$$K_X \sim_{\mathbf{Q}} f^*(K_Y + B_Y + M_Y) - \Delta.$$

Assume either of the 2 condition holds:

1. $K_Y + (1-\epsilon)B_Y$ peff for some $\epsilon > 0$, or
2. The class $\{K_F\}$ is *rigid* for a general fibre.

Then $K_Y + B_Y + M_Y$ is big.

First Idea to handle Δ

- Take m_0 and m_1 sufficiently divisible and $m_1 \gg m_0$. Consider the Bergman-kernel metric φ_1 on

$$L_1 = m_1 K_{X/Y} + m_0 K_X - f^* H$$

- The Bergman-kernel metric is super explicit when $\kappa(F) = 0$.
- Use Siu decomposition on L_1, i.e.
- If $T_1 = dd^c \varphi_1$, then $L_1 - \sum_W \nu_W(T_1)[W]$ is still pseudo-effective, where $\nu_W(T_1)$ is the generic Lelong number of T_1 along W (analytic analogue for multiplicity).
- We have a precise control on the Lelong numbers along some components of Δ.

First idea: Bergman-kernel metric I

- From the horizontal divisors: $L_1 - (m_1 + m_0)\Delta^h$ peff, essentially from the definition of the Bergman-kernel metric.
- This immediately recovers Schnell's assumption since if K_Y peff, then

$$\left(m_1 K_{X/Y} + m_0 K_X - f^* H - (m_1 + m_0)\Delta^h\right) + m_1 K_Y$$

is peff, and this descends to

$$(m_1 + m_0)(K_Y + B_Y + M_Y) - f^* H$$

being peff.

First idea: Bergman-kernel metric II

- Upshot: We can subtract more from the vertical divisors.
- Conclusion:

$$m_1 K_{X/Y} + m_0 K_X - f^*H - (m_1 + m_0)\Delta^h - \sum_E \alpha_E E$$

  is peff, where E runs through the vertical divisors in the singular fibres.
- α_E is related to certain volume asymptotics, which is well understood due to Takayama, or Boucksom–Jonsson
- We only have to assume that $K_Y + (1-\epsilon)B_Y$ peff, instead K_Y being pseff.
- Unfortunately, this is not always the case (semi-stable family of elliptic curves with large variation)

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- This leads us to to the following inductive approach:

$$\begin{aligned} &\text{Non-vanishing in dimension } n \\ &+ \text{Rigidity of } K_X \text{ (for } \kappa(X) = 0\text{) in dimension } n-1 \\ &\Longrightarrow \text{ Campana–Peternell conjecture in dimension } n. \end{aligned}$$

- Unconditional result for 4-folds.

Thank you