Abundance for Kähler Varieties via Algebraic Reduction

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Bimeromorphic Geometry

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A Kähler class plays a similar role in Kähler geometry as an ample class does in projective geometry, which endows the Kähler manifolds with numerical properties of positivity.

Bimeromorphic Geometry

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- To fix this, we introduce 'transcendental curves', i.e. positive $(1,1)$-currents. Let $\overline{\mathrm{NA}}(X)$ be the closed cone generated by positive $(1,1)$-currents on X. Then

Theorem

$\overline{\mathrm{NA}}(X)$ *and* $\mathrm{Nef}(X)$ *are dual to each other.*

The MMP for Kähler Varieties

One breakthrough in bimeromorphic geometry is the establishment of the minimal model program for Kähler threefolds:

Theorem (Höring, Peternell 2016)

Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with terminal singularities. If K_X is pseudoeffective, then X has a minimal model.

Theorem (Das, Hacon 2022)

Let (X, B) be a dlt pair where X is a $\mathbb{Q}$-factorial compact Kähler 3-fold. If $K_X + B$ is pseudo-effective, then there exists a finite sequence of flips and divisorial contractions $\phi : X \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n$ such that $K_{X_n} + \phi_* B$ is nef.

Abundance for Kähler Varieties

Then it is natural to ask if the abundance holds:

Conjecture

Let (X, Δ) be a Kähler lc pair. If $K_X + \Delta$ is nef, then $|m(K_X + \Delta)|$ is base point free for sufficiently divisible $m \in \mathbb{N}$.

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In dimension 3, abundance and log abundance for Kähler varieties are established by Campana, Höring, Peternell, Das and Ou.

Abundance for Kähler Varieties

We use a different approach on the abundance problem. Our method can recover the abundance for Kähler threefold in the case $a(X) \neq 0$:

Theorem

Let (X, Δ) be a klt $\mathbb{Q}$-factorial Kähler threefold with $a(X) \neq 0$. If $K_X + \Delta$ is nef, then $K_X + \Delta$ is semiample.

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Our idea is to reduce abundance for Kähler varieties to abundance for projective varieties via the **algebraic reduction map**.

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In other words, focusing only on the meromorphic functions, X looks like an algebraic variety of dimension $a(X)$.
Indeed there is a natural map, called the **algebraic reduction map**:

$$X \dashrightarrow V$$

such that V is projective and $\mathbb{C}(X) = \mathbb{C}(V)$.

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Reduce the abundance for the Kähler variety X to the abundance for a projective variety along the algebraic reduction fibration $X \dashrightarrow V$.

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- Problem 1: The algebraic reduction map is not defined everywhere. After resolution of the indeterminacy, the new canonical divisor on the higher model is no longer nef.
- Problem 2: We need a theory to compare the canonical divisors, K_Y and K_V, along the algebraic reduction map.

Once these problems get solved, one can follow Florin Ambro's argument to obtain abundance on X.

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For a divisor D on X, a **Zariski decomposition** (T. Fujita 1984) is a decomposition of D on a model $\pi : Y \to X$:

$$\pi^* D = P + N$$

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The Canonical Bundle Formula

- We need a theory to compare the canonical divisors K_Y and K_V along the algebraic reduction map.

The answer is the canonical bundle formula. For a K-trivial fibration $f : (X, \Delta) \to Y$, the canonical bundle formula takes the form:

$$K_X + \Delta \sim_{\mathbb{Q}} f^*(K_V + B_V + M_V)$$

- The **discriminant divisor** B_V describes singularities of the fibers. Roughly speaking, it measures how far a fiber is from being log canonical.
- The **moduli divisor** M_V describes the variation of the fibers. So it is natural to expect M_V to be 'positive' in some sense.
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To realize this idea, there turns out to be two problems:

- Problem 1: The algebraic reduction map is not defined everywhere. After resolution of the indeterminacy, the new canonical divisor on the higher model is no longer nef. (Zariski decomposition theory)
- Problem 2: We need a theory to compare the canonical divisors, K_Y and K_V, along the algebraic reduction map. (Canonical Bundle Formula)

Higher Dimension

We want to generalize the argument to higher dimensions. Let X be a Kähler variety. Consider the algebraic reduction fibration

$$F \to X \dashrightarrow V$$

- Roughly speaking, we want to use the same argument to show if abundance holds for F, V, then it holds for X.
- On the one hand, the base variety V is projective. On the other hand, though the fiber F might be analytic, it has lower dimension which is suitable for an inductive argument.

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- One obstacle of this approach is that we have no control of the locus of indeterminacy.
  We would like to ask the following question:

Almost Holomorphicity

Definition

Let $f : X \dashrightarrow Y$ be a meromorphic map between normal compact varieties. Let $X^0 \subseteq X$ be the maximal open subset where f is holomorphic. The map f is said to be ***almost holomorphic*** *if some fibers of the restriction $f|_{X^0}$ are compact.*

Conjecture

Let X be a compact Kähler manifold. Then the algebraic reduction map is almost holomorphic.

It is known for threefolds and some special cases in higher dimensions.

Thank you!