Special cycles and cones of divisors on orthogonal modular varieties

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Overview

1. **Special cycles on orthogonal modular varieties**

2. **Uniruledness**

3. **Special cycles under "tautological maps"**

4. **Cones of divisors**

Orthogonal modular varieties

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K3 surfaces

$$\Lambda = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus \mathbb{Z}\ell, \quad \text{with} \quad \langle \ell, \ell \rangle = -2d$$

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Hyperkähler manifolds

For various choices of Λ

$$\mathcal{F}_\Lambda = \text{(finite cover of) partial compactification of moduli of HK manifolds}$$

Noether–Lefschetz divisors/Heegner divisors on $\mathcal{F}_\Lambda$

In K3 setting: Noether–Lefschetz divisors

$$D_{h,a} = \{(S, H) \mid \exists \beta \in \mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_\Lambda) \text{ s.t. } \beta^2 = 2h - 2, \beta.H = a\}.$$

Examples

- $D_{0,0}$ = nodal locus
  (there is a (-2)-curve β s.t. $\beta.H = 0$)
- $D_{1,1}$=unigonal locus
  (there is a curve β with $\beta^2 = 0$ and $\beta.H = 1$)

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In general setting: Heegner divisors

For fixed $v \in \Lambda^\vee \subset \Lambda_\mathbb{Q}$

$$D_v = v^\perp \cap \mathcal{D}_\Lambda = \{[Z] \in \mathcal{D}_\Lambda \,|\langle Z, v\rangle = 0\}\,.$$

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$$D_v = v^\perp \cap \mathcal{D}_\Lambda = \{[Z] \in \mathcal{D}_\Lambda \,|\, \langle Z, v\rangle = 0\}\,.$$

Let $\ell + \Lambda \in D(\Lambda) = \Lambda^\vee/\Lambda$ and $m \in \frac{\langle \ell,\ell\rangle}{2} + \mathbb{Z}$ non-positive, then get $\widetilde{\mathrm{O}}^+(\Lambda)$-invariant cycle

$$\sum_{\substack{v \in \ell + \Lambda \\ \frac{\langle v,v\rangle}{2} = m}} D_v \tag{1}$$

$\Longrightarrow$ descends to a $\mathbb{Q}$-Cartier divisor $H_{m,\ell}$ on $\mathcal{F}_\Lambda = \mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)$ called a Heegner divisor.

Heegner divisors generate $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{\Lambda})$

Theorem (Bergeron–Li–Millson–Moeglin)

Let Λ be an even lattice of signature $(2, n)$ with $n \geq 3$ splitting off two copies of the hyperbolic plane. Then

$$\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{\Lambda}) = \langle \text{Heegner divisors} \rangle.$$

In particular, in the K3 setting this says

$$\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{\Lambda}) = \langle \text{Noether–Lefschetz divisors} \rangle.$$

Relationship to modular forms

- The *metaplectic group* $\mathrm{Mp}_2(\mathbb{Z})$ is a double cover of $\mathrm{SL}_2(\mathbb{Z})$ of pairs $(A, \phi(\tau))$ where $A \in \mathrm{SL}_2(\mathbb{Z})$, $\phi(\tau)$ a choice of a square root $c\tau + d$ on $\mathbb{H}$.

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- The *Weil representation* of $\mathrm{Mp}_2(\mathbb{Z})$ attached to Λ is a canonical representation

$$\rho_\Lambda : \mathrm{Mp}_2(\mathbb{Z}) \longrightarrow \mathrm{GL}\left(\mathbb{C}\left[D(\Lambda)\right]\right).$$

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Vector-valued modular forms

A holomorphic function

$$f : \mathbb{H} \longrightarrow \mathbb{C}\left[D(\Lambda)\right]$$

is a vector-valued (elliptic) modular form of weight k and type ρ_Λ if for all $g = (A, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$

$$f(A\tau) = \phi(\tau)^{2k}\rho_\Lambda(g) \cdot f(\tau)$$

and f is holomorphic at the cusp at ∞.

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- $M_\Lambda^k = \mathbb{Q}$-vector space of vector-valued modular forms of weight k and type ρ_Λ
- Elements $f \in M_\Lambda^k$ have Fourier expansions

$$f = \sum_{\ell \in D(\Lambda)} \sum_{\substack{m \in q(\ell)+\mathbb{Z} \\ m \geq 0}} a_{m,\ell} q^m \mathfrak{e}_\ell,$$

where $a_{m,\ell} \in \mathbb{Q}$, $q^m = \exp(2\pi i m\tau)$, $\tau \in \mathbb{H}$, and $\mathfrak{e}_\ell$ standard generators of $\mathbb{C}[D(\Lambda)]$.

Relationship to modular forms

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Theorem (Kudla–Millson)

$$\sum_{\ell \in D(\Lambda)} \sum_{\substack{m \in q(\ell)+\mathbb{Z} \\ m \geq 0}} [H_{m,\ell}] q^m \mathfrak{e}_\ell \in M_\Lambda^k \otimes H^2(\mathcal{F}_\Lambda, \mathbb{Q}) \tag{2}$$

The $[H_{m,\ell}]$ are cohomology classes of Heegner divisors.

Relationship to modular forms

- M^k_Λ = $\mathbb{Q}$-vector space of vector-valued modular forms of weight k and type ρ_Λ

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The $[H_{m,\ell}]$ are cohomology classes of Heegner divisors.

- Equivalently, for every functional $F \in (H^2(\mathcal{F}_\Lambda, \mathbb{Q}))^\vee$, the series

$$\sum_{\ell \in D(\Lambda)} \sum_{\substack{m \in q(\ell)+\mathbb{Z} \\ m \leq 0}} F([H_{m,\ell}]) q^m \mathfrak{e}_\ell$$

converges and is a modular form in M^k_Λ.

Relationship to modular forms

- $M_\Lambda^k = \mathbb{Q}$-vector space of vector-valued modular forms of weight k and type ρ_Λ
- Kudla–Millson implies there is a map

$$\begin{aligned}(H^2(\mathcal{F}_\Lambda, \mathbb{Q}))^\vee &\to M_\Lambda^k \\ F &\mapsto \sum_{\ell \in D(\Lambda)} \sum_{\substack{m \in q(\ell)+\mathbb{Z} \\ m \leq 0}} F([H_{m,\ell}]) q^m \mathfrak{e}_\ell .\end{aligned}$$

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- Dualizing yields

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\begin{aligned}
\psi \colon \left(M_\Lambda^k\right)^\vee &\to H^2(\mathcal{F}_\Lambda, \mathbb{Q}) \\
c_{m,\ell} &\mapsto [H_{-m,\ell}]
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Relationship to modular forms

UPSHOT: There is a map of $\mathbb{Q}$-vector spaces

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Remark

The Bergeron–Li–Millson–Moeglin result that (for Λ splitting 2 copies of U)

$$\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_\Lambda) = \langle H_{m,\ell} \rangle$$

in fact is showing that $\psi\colon \left(M_\Lambda^{k,\circ}\right)^\vee \to H^2(\mathcal{F}_\Lambda, \mathbb{Q})$ is an isomorphism, where $k = \frac{n+2}{2}$.

(Injectivity already shown by Bruinier).

Higher codimension: special cycles

- $\mathcal{S}_g$ = symmetric half-integral $g \times g$-matrices (diagonal entries integer).
- For $v = (v_1, \dots, v_g) \in (\Lambda^\vee)^g$, denote by $v^\perp$ the set of points in $\mathcal{D}_\Lambda$ which are orthogonal to every entry of v.
- The *moment matrix* $q(v)$ of v is $q(v) = \frac{1}{2}\big(\langle v_a, v_b\rangle\big)_{a,b=1,\dots,g}$. Note that if $v \in \Lambda^g$, then $q(v) \in \mathcal{S}_g$.

Special Cycles

Let $\ell \in (D(L))^g$ and let $T \in q(\ell) + \mathcal{S}_g$. The formal sum

$$\sum_{\substack{v \in \ell + \Lambda^g \\ q(v) = T}} v^\perp$$

is a locally finite $\widetilde{\mathrm{O}}^+(\Lambda)$-invariant cycle in $\mathcal{D}_\Lambda$.

$\implies$ Descends to a codimension g special cycle $Z(T, \ell)$ on $\mathcal{F}_\Lambda$.

Higher codimension: modular forms

- $\mathbb{H}_g$ = Siegel upper half-space of genus g, $\mathrm{Sp}_{2g}(\mathbb{R})$ acts on $\mathbb{H}_g$
- $\rho_{\Lambda,g}$ =Weil rep. of the metaplectic cover $\mathrm{Mp}_{2g}(\mathbb{Z})$ of $\mathrm{Sp}_{2g}(\mathbb{Z})$ acting on $\mathbb{C}[D(\Lambda)^g]$

Definition: Higher genus modular forms

A vector-valued (Siegel) modular form of weight k and genus g with respect to the Weil representation $\rho_{\Lambda,g}$ is a holomorphic function $f\colon \mathbb{H}_g \to \mathbb{C}[D(\Lambda)^g]$ such that

$$f(\gamma \cdot \tau) = \phi(\tau)^{2k} \rho_{\Lambda,g}(\gamma) f(\tau) \qquad \text{for all } \gamma = (M, \phi) \in \mathrm{Mp}_{2g}(\mathbb{Z}).$$

If $g = 1$, then we also require that f is holomorphic at ∞.

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- $M^k_{g,\Lambda}$ = $\mathbb{Q}$-vector space of such modular forms.
- Every $f \in M^k_{g,\Lambda}$ admits a Fourier expansion of the form

$$f(\tau) = \sum_{\ell \in D^g_\Lambda} \sum_{\substack{T \in q(\ell)+\mathcal{S}_g \\ T \leq 0}} a_{T,\ell}(f) q^T \mathfrak{e}_\ell, \qquad \text{where } q^T = e^{2\pi i \operatorname{tr}(T\tau)}.$$

Higher codimension

Theorem (Kudla–Millson)

$$\sum_{\ell \in D_\Lambda^g} \sum_{\substack{T \in q(\ell) + \mathcal{S}_g \\ T \leq 0}} [Z(T, \ell)] \cdot [-\omega]^{g - \mathrm{rk}\, T} q^T \mathfrak{e}_\ell \in M_{g,\Lambda}^k \otimes H^{2g}(\mathcal{F}_\Lambda, \mathbb{Q})$$

- As before, get a map

$$\begin{aligned}(H^{2g}(\mathcal{F}_\Lambda, \mathbb{Q}))^\vee &\to M_{g,\Lambda}^k \\ F &\mapsto \sum_{\ell \in D_\Lambda^g} \sum_{\substack{T \in q(\ell) + \mathcal{S}_g \\ T \leq 0}} F\left([Z(T, \ell)]\right) \cdot [-\omega]^{g - \mathrm{rk}\, T} q^T \mathfrak{e}_\ell.\end{aligned}$$

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- Dualizing we have

$$\psi \colon \left(M_{g,\Lambda}^k\right)^\vee \to H^{2g}(\mathcal{F}_\Lambda, \mathbb{Q})$$
$$c_{-T,\ell} \mapsto [Z(T,\ell)]$$

Main Goal

UPSHOT

There is a map of $\mathbb{Q}$-vector spaces

$$\psi\colon \left(M^k_{g,\Lambda}\right)^\vee \to H^{2g}(\mathcal{F}_\Lambda, \mathbb{Q})$$

If Λ splits off $U^{\oplus 2}$ and $n \geq 3$ (the geometric setting), get an isomorphism

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- Special cycles on $\mathcal{F}_\Lambda$ under "tautological maps"?
- Effective and extremal divisors on $\mathcal{F}_\Lambda$?

Uniruledness of $\mathcal{F}_\Lambda$

Let Λ be an even lattice of signature $(2, n)$ with $n \geq 3$ splitting off two copies of U.
For $k = \frac{n+2}{2}$

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In $M^{k,\circ}_{1,\Lambda}$ is the Eisenstein series

$$E_{k,\Lambda}(\tau) = \sum_{(A,\phi)\in\widetilde{\Gamma}_\infty\backslash\mathrm{Mp}_2(\mathbb{Z})} \phi(\tau)^{2k} \cdot \rho_\Lambda(A,\phi)^{-1}\mathfrak{e}_0 = \sum_{m,\ell} e_{m,\ell}q^m\mathfrak{e}_\ell.$$

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Proposition (Barros–Beri–F.–Williams '24)

The orthogonal modular variety $\mathcal{F}_\Lambda$ is uniruled if

$$n e_{0,0} + \frac{1}{4} e_{1,0} < 0.$$

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- Choose curve class $C = \lambda^{n-1}$ to not meet boundary of $\mathcal{F}_\Lambda^{BB}$ (hence same for $\mathcal{F}_\Lambda^{\mathrm{tor}}$).

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- By work of Kudla this series (3) is equal to $\gamma E_{k,\Lambda}(\tau)$ for some $\gamma > 0$

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- So if $ne_{0,0} + \frac{1}{4}e_{1,0} < 0$, then $K_Y.C < 0$ and so Y and thus $\mathcal{F}_\Lambda$ is uniruled.

Uniruledness for some moduli of hyperkähler manifolds

Proposition (Barros–Beri–F.–Williams '24)

The orthogonal modular variety $\mathcal{F}_\Lambda$ is uniruled if $ne_{0,0} + \frac{1}{4}e_{1,0} < 0$.

Corollary

The moduli space $\mathcal{M}^{\gamma}_{\mathrm{OG6},2d}$ is uniruled in the following cases

(i) when $\gamma = 1$ for $d \leq 12$,

(ii) when $\gamma = 2$ for $t \leq 10$ and $t = 12$ with $d = 4t - 1$,

(iii) when $\gamma = 2$ for $t \leq 9$ and $t = 11, 13$ with $d = 4t - 2$.

The moduli spaces $\mathcal{M}^{1}_{\mathrm{Kum}_n,2}$ and $\mathcal{M}^{2}_{\mathrm{Kum}_n,2}$ are uniruled in the following cases:

(i) when $\gamma = 1$ for $n \leq 15$ and $n = 17, 20$,

(ii) when $\gamma = 2$ for $t \leq 11$ and $t = 13, 15, 17, 19$, where $n = 4t - 2$.

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Question

Describe pullback of special cycles?

Special cycles under pullback

Let $L \subset \Lambda$ sublattice of signature $(2, n')$ with $n' \leq n$. Let $K = L^{\perp}$ and $\Omega = L \oplus K$.

Theorem (Barros–F.–Zuffetti) (forthcoming '25)

The following diagram commutes

$$\begin{array}{ccc} (M_{g,\Lambda}^{\mathrm{rk}\,\Lambda/2})^{\vee} & \xrightarrow{(\Theta_K)^{\vee} \circ \mathrm{tr}^{\vee}_{\Lambda/\Omega}} & (M_{g,L}^{\mathrm{rk}\,L/2})^{\vee} \\ \downarrow \psi_{\Lambda} & & \downarrow \psi_L \\ H^2(\mathcal{F}_{\Lambda}, \mathbb{C}) & \xrightarrow{\varphi^*} & H^2(\mathcal{F}_L, \mathbb{C}) \end{array} \tag{4}$$

In the above, $\mathrm{tr}_{g,\Lambda/\Omega} \colon M^k_{g,\Omega} \longrightarrow M^k_{g,\Lambda}$ is the trace map and $\Theta_{K,g}(\tau)$ is the vector-valued genus g theta function in $M_{g,K}^{\mathrm{rk}\,K/2}$ arising from K.

Special cycles under pullback

Corollary (Barros–F.–Zuffetti) (forthcoming '25)

For the codimension g special cycle $Z(T,\bar{\ell})^\Lambda$ on $\mathcal{F}_\Lambda$, where $\bar{\ell} \in D(\Lambda)^g$, $T \in q(\bar{\ell}) + \mathcal{S}_g$, we have

$$\varphi^*([Z(T,\bar{\ell})^\Lambda]) = \sum_{\alpha \in (\Lambda/\Omega)^g} \sum_{\substack{t \in q((\alpha+\ell)_L)+\mathcal{S}_g \\ t \geq 0}} \theta_g(T-t,(\alpha+\ell)_K)[Z(t,(\alpha+\ell)_L)^L] \cup [-\omega]^{g-\mathrm{rk}\, t},$$

where $\ell \in \Lambda^\vee/\Omega$ is any fixed preimage of $\bar{\ell}$ under $\Lambda^\vee/\Omega \to \Lambda^\vee/\Lambda$ and $\theta_g(T,\kappa)$ the Fourier coefficient of index (T,κ) of $\Theta_{K,g}$.

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POINT: We obtain an explicit description of the pullback of a special cycle.

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For Λ splitting off $U^{\oplus 2}$ and $n \geq 3$, we give an explicit list of generators of $\mathrm{Eff}^{NL}(\mathcal{F}_\Lambda)$.

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Example

In the K3 setting, where

$$\Lambda = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus \mathbb{Z}\ell, \quad \text{with} \quad \langle \ell, \ell \rangle = -2d$$

$$\mathcal{F}_\Lambda = \text{moduli space of quasi-polarized K3 surfaces of degree } 2d$$

we write down $\mathrm{Eff}^{NL}(\mathcal{F}_\Lambda)$ for $d \leq 20$ (e.g. $\mathrm{Eff}^{NL}(\mathcal{F}_2) = \langle D_{0,0} - D_{1,1}, D_{1,1} \rangle$).

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$\implies$ translate to a problem about modular forms.

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$$-\sum_{\substack{\alpha \in \Lambda / L \oplus K \\ 0 < t < -m}} e_{t,\alpha_L} < \frac{1}{2}$$

for $E_{\frac{n+1}{2},L}(\tau) = \sum_{m,\ell} e_{m,\ell} q^m \mathfrak{e}_\ell$ the weight $\frac{\mathrm{rk}\, L}{2}$ Eisenstein series for L, then the irreducible component P^Λ_{m,ρ_*} of H^Λ_{m,ρ_*} is extremal.

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- Use pullback formula for special cycles to compute $\varphi^* H_{-m,\ell}$ and similar tricks as in uniruledness proof to get $\varphi^* C \cdot \varphi^* H_{-m,\ell} < 0$.
- Projection formula implies $C.H_{-m,\ell} < 0$.

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Example

In the K3 setting, for low degrees $d \leq 7$, this shows that all generators of $\mathrm{Eff}^{NL}_{\mathbb{R}}(\mathcal{F}_\Lambda)$ except the nodal Noether–Lefschetz divisor $D_{0,0}$ are extremal in $\overline{\mathrm{Eff}}(\mathcal{F}_\Lambda)$.

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Corollary (Barros–F.–Zuffetti)

The orthogonal modular variety $\mathcal{F}_\Lambda$ with $\Lambda = U^{\oplus 2} \oplus A_1(-1) \oplus A_1(-3)$ partially compactifying the moduli space $\mathcal{M}^1_{\mathrm{Kum}_2,2}$ satisfies

$$\mathrm{Eff}^{NL}_{\mathbb{R}}(\mathcal{F}_\Lambda) = \overline{\mathrm{Eff}}(\mathcal{F}_\Lambda).$$

The End