Boundedness of abelian fibrations

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(joint with S. Filipazzi, F. Greer, M. Mauri, & R. Svaldi)

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Introduction: Boundedness

Definition

A collection $\{X_i\}_{i\in I}$ of projective varieties is *bounded* if there exists a family of projective varieties $\mathcal{X} \to \mathcal{T}$, over a quasiprojective base $\mathcal{T}$, such that for all $i \in I$, there exists a $t \in \mathcal{T}$ for which $X_i \simeq \mathcal{X}_t$.

Similar notions:

1. Birational boundedness
2. Bounded in codimension one
3. Boundedness of pairs
4. Analytic boundedness
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Examples:

1. Subschemes of $\mathbb{P}^n$ with a fixed Hilbert polynomial
2. Smooth Fano varieties of a fixed dimension
3. K3 and abelian surfaces: Analytically bounded, but algebraically unbounded (moduli of K3 surfaces $\mathcal{F}_{2d}$ for any degree $2d > 0$)

What about K-trivial varieties of higher dimension?

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What about K-trivial varieties of higher dimension?

Introduction: K-trivial varieties

Definition

A *K-trivial variety* is a normal projective variety X with canonical singularities such that $K_X \sim 0$. We say that X is

- (CY) *Calabi-Yau* if $H^0(X, \Omega^{[k]}) = 0$ for all $0 < k < \dim X \geq 3$;
- (ICY) *irreducible Calabi-Yau* if all quasi-étale covers of X are CY;
- (PS) *primitive symplectic* if $H^0(X, \Omega^{[1]}) = 0$, $H^0(X, \Omega^{[2]}) = \mathbb{C}\sigma$ and σ is symplectic on the smooth locus of X;
- (IS) *irreducible symplectic* if all quasi-étale covers of X are PS;
- (AV) an *abelian variety* if $H^0(X, \Omega^{[1]}) = \dim X$.

Theorem (Beauville-Bogomolov decomposition)

Every variety with numerically trivial canonical bundle and klt singularities admits a quasi-étale cover which is a product of ICY varieties, IS varieties, and an abelian variety.

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Smooth hypersurfaces of degree $d+1$ in $\mathbb{P}^d$ are irreducible Calabi-Yau varieties.

Example (Alexeev)

Let G be a finite group acting on a lattice L and suppose $L_{\mathbb{C}}$ is an irreducible representation of G. Then, for any abelian surface A,

$$X := G \backslash L \otimes A$$

is a primitive symplectic variety, but is not irreducible symplectic; X has a quasi-étale cover by the abelian variety $L \otimes A \simeq A^{\oplus \operatorname{rk} L}$.

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Are Calabi-Yau varieties of a fixed dimension bounded? Are K-trivial varieties of a fixed dimension analytically bounded?

This is a famous and very difficult problem; we don't have techniques to attack the general question.

Definition

A *fibration* $f : X \to Y$ is a surjective, proper morphism of normal varieties with connected fibers and $0 < \dim Y < \dim X$.

If $K_X \sim 0$, by adjunction, the general fiber of f is also K-trivial.

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Main Results

Theorem ICY (E.-Filipazzi-Greer-Mauri-Svaldi)

1. *Abelian-fibered irreducible Calabi-Yau varieties* X*, of a fixed dimension, are birationally bounded.*
2. *Primitive symplectic-fibered irreducible Calabi-Yau varieties* X*, of a fixed dimension and fibered in a fixed analytic deformation class, are birationally bounded.*

A corollary: fibered ICY 3-folds are bounded (here it is not hard to pass from birational boundedness to boundedness).

Theorem PS (E.-Filipazzi-Greer-Mauri-Svaldi)

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Main results: Some consequences

Conjecture (HyperKähler SYZ/Generalized Abundance)

Let X be a primitive symplectic variety of dimension $2d$, which admits a nontrivial nef line bundle $L \to X$ for which $L^{2d} = 0$. Then X admits a Lagrangian fibration.

By our second theorem:

Corollary

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Canonical bundle formula

Let $f : X \to Y$ be a fibration in K-trivial varieties (with $K_X \sim_f 0$). The base admits the structure of a *generalized pair* $(Y, B, \mathbf{M})$. An effective $\mathbb{Q}$-divisor (the *boundary divisor*)

$$B := \sum_{\substack{P \subset Y \text{ prime} \\ \text{divisors}}} a_P P$$

measures singularities of X over the codimension 1 points of Y, while the *moduli divisor*

$$\mathbf{M} := c_1(\mathcal{H}^{g,0})$$

is the class of the $\mathbb{Q}$-line bundle formed from the $(g, 0)$-part of the Hodge structures on the fibers of f ($g =$ the fiber dimension).

Theorem (Canonical bundle formula)

$$K_X \sim f^*(K_Y + B + \mathbf{M})$$

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Canonical bundle formula: Example

Let $f : X \to Y$ be a relatively minimal elliptic surface. Then

$$K_X \sim f^*(K_Y + B + j^*\mathcal{O}(\tfrac{1}{12}))$$

where $j : Y \to \mathbb{P}^1$ is the j-invariant, and $B = \sum a_P P$ and $a_P \in [0,1)$ depends on the Kodaira type of the fiber:

$f^{-1}(P)$	$I_n(m)$	II	III	IV	I_n^*	II^*	III^*	IV^*
a_P	$\frac{m-1}{m}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{2}{3}$

Proof: Key steps

We outline the proof of birational boundedness of abelian fibrations $f : X \to Y$ of ICY varieties.

tep 1: Bound the integer $c > 0$ for which $c\mathbf{M}$ is b-free.

tep 2: Bound the possible bases $(Y, B, \mathbf{M})$ in codimension 1.

tep 3: Bound (birationally) the Albanese fibration $f^{\mathrm{Alb}} : X^{\mathrm{Alb}} \to Y$ of any $f : X \to Y$ inducing $(Y, B, \mathbf{M})$.

tep 4: Bound the *Tate–Shafarevich group*, of abelian fibrations $f : X \to Y$ with a fixed Albanese and which induce $(Y, B, \mathbf{M})$.

See also: Gross, birationally bounding elliptic CY 3-folds.

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Proof: Step 1 (effective b-semiampleness)

Let $f : X \to Y$ be a fibration in K-trivial varieties. Then $\mathbf{M}$ is b-nef, and conjecturally it is also b-semiample ("b-semiampleness conjecture" of Prokhorov and Shokurov).

Relatedly, Laza conjectures that all moduli spaces $\mathcal{M}$ of K-trivial varieties admit a "Baily-Borel" compactification $\mathcal{M} \hookrightarrow \overline{\mathcal{M}}$ on which the moduli divisor λ of the universal family is ample.

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For abelian and primitive symplectic varieties, the moduli space $\mathcal{M} = \Gamma\backslash\mathbb{D}$ is locally Hermitian symmetric and the Baily-Borel compactification $\overline{\Gamma\backslash\mathbb{D}}$ exists, by the Baily-Borel theorem.

Given an abelian (or primitive symplectic) fibration $f : X \to Y$, there is an induced period map

$$\Phi : Y \dashrightarrow \overline{\Gamma\backslash\mathbb{D}}.$$

Thus, $c\mathbf{M}$, $c > 0$ is b-free once the universal moduli divisor $c\lambda$ is free. The issue: moduli spaces of abelian g-folds are not finite in number. For all sequences $\mathbf{d}$ of integers $d_1 \mid \cdots \mid d_g$ we have a DM stack $\mathcal{A}_{g,\mathbf{d}}$ of $\mathbf{d}$-polarized abelian g-folds.

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Proof: Completion of Step 1 (effective b-semiampleness)

Answer: The "Zarhin trick." All the Baily-Borel compactifications map into a single moduli space

$$\overline{A}_{g,\mathbf{d}} \to \overline{A}_{8g}$$

of PPAVs, and the Hodge bundle λ_{8g} pulls back to $8\lambda_g$ on each $\overline{A}_{g,\mathbf{d}}$ (critically, independent of $\mathbf{d}$).

Proof: The Zarhin trick sends $Zar : A \mapsto A^{\oplus 4} \oplus (A^*)^{\oplus 4}$ and so

$$H^{8g,0}(Zar(A)) \simeq H^{g,0}(A)^{\otimes 8}.$$

(For families of primitive symplectic varieties, we first apply the Kuga-Satake construction, then we use the Zarhin trick.)

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Proof: Step 2 (bounding the base)

So in a fixed dimension, we bounded the integer c for which $c\mathbf{M}$ is b-free. Birkar–di Cerbo–Svaldi imples the bases $(Y, B, \mathbf{M})$ are bounded in codimension 1, when Y is rationally connected.

When X is ICY, this is true. So there is a finite type family

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which contain all possible bases (up to small modification) of an abelian fibration $f : X \to Y$ of an ICY variety.

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Proof: Step 3 (bounding the Albanese fibration)

Our next goal (crucial): Bound the polarization type $\mathbf{d}$ of the fibers, then the classifying morphism

$$\Phi : Y \dashrightarrow \mathcal{A}_{g,\mathbf{d}}$$
$$y \mapsto \mathrm{Aut}^0(X_y)$$

Warning: In general, $f : X \to Y$ is not the pullback of the universal family $\mathcal{X}_{g,\mathbf{d}} \to \mathcal{A}_{g,\mathbf{d}}$ along the classifying morphism!! The stack $\mathcal{A}_{g,\mathbf{d}}$ classifies abelian varieties *with a distinguished origin*.

Given $f : X \to Y$, we define a birational class of abelian fibration $f^{\mathrm{Alb}} : X^{\mathrm{Alb}} \to Y$ whose fiber over $y \in Y$ is the group of translations $\mathrm{Aut}^0(X_y)$—this is the pullback of the universal family along Φ.

X is birational to X^{Alb} iff X admits a rational section.

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Proof: Step 3, bounding the Albanese fibration

Again use Zarhin: By a volume argument, rational maps

$$Zar \circ \Phi : Y \dashrightarrow \overline{A}_{8g}$$

for which $(Zar \circ \Phi)^*(\lambda_{8g}) \equiv \mathbf{M}$ are bounded. So the space of all possible "Zarhin-tricked" period maps $(Y, Zar \circ \Phi)$ is bounded.

Question: **Can we undo the Zarhin trick and in turn bound the original period map $\Phi : Y \dashrightarrow \overline{A}_{g,\mathbf{d}}$?**

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In general, no! For all $A \in \mathcal{A}_{g,\mathbf{d}}$ we have a (very stupid) abelian fibration $A \to *$. The Zarhin-tricked period maps $* \to \overline{A}_{8g}$ lie in a bounded family. But the original collection of maps is unbounded, since we may take $\mathbf{d}$ arbitrary. To resolve this difficulty:

Lemma

If $h^2(X, \mathcal{O}) = 0$, the $Zar \circ \Phi$ pullback of the universal $\mathbb{Z}^{16g}$-local system on $\mathcal{A}_{8g}$ to the smooth locus $Y^o \subset Y$ of f recovers a polarization type $\mathbf{d}$.

Proof.

Deligne's theorem of the fixed part. □

(Note: This lemma holds for a Lagrangian fibration $f : X \to Y$, which is key to proving Theorem PS.)

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Lemma

If $h^2(X, \mathcal{O}) = 0$, the $Zar \circ \Phi$ pullback of the universal $\mathbb{Z}^{16g}$-local system on $\mathcal{A}_{8g}$ to the smooth locus $Y^o \subset Y$ of f recovers a polarization type $\mathbf{d}$.

Proof.

Deligne's theorem of the fixed part. □

(Note: This lemma holds for a Lagrangian fibration $f : X \to Y$, which is key to proving Theorem PS.)

Proof: Step 3 (bounding the Albanese fibration)

Thus, we have bounded the polarization type $\mathbf{d}$ of a general fiber of $f : X \to Y$. Repeating the earlier volume argument, we bound the classifying morphism

$$\Phi^o : Y^o \to \mathcal{A}_{g,\mathbf{d}}$$

and in turn the birational class of the Albanese $f^{\mathrm{Alb}} : X^{\mathrm{Alb}} \to Y$.

(The role of the boundary divisor B is very subtle—it is critical to make a distinction between the period map to the coarse space $Y \dashrightarrow A_{g,\mathbf{d}}$ and the classifying map to the DM stack $Y^o \to \mathcal{A}_{g,\mathbf{d}}$. Lifts from the coarse space to the stack are controlled by the topology of the complement of $\operatorname{supp} B$.)

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Proof: Step 4 (bounding the Tate–Shafarevich group)

This step is very technical (≈ 20-30 pages). Question: How to bound $f : X \to Y$ from the data of $f^{\mathrm{Alb}} : X^{\mathrm{Alb}} \to Y$?

Set $Y^+ :=$ the **big** open subset of Y^{reg} where the discriminant of f is smooth, divisorial. Find a finite Galois G-cover $\widetilde{Y}^+ \to Y^+$ depending only on $(X^{\mathrm{Alb}} \to Y, B, \mathbf{M})$ for which we have a key diagram:

Here $\widetilde{f}^+ : \widetilde{X}^+ \to \widetilde{Y}^+$ is the normalized base change of the restriction $f^+ : X^+ \to Y^+$ and $(\widetilde{f}^+)^{\mathrm{Alb}}$ is a G-equivariant Kulikov model of the Albanese fibration of $\widetilde{X}^+ \to \widetilde{Y}^+$, to which $\widetilde{f}^+$ is étale-locally birational.

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$$\begin{array}{ccccc} X^+ & \xleftarrow{/G} & \widetilde{X}^+ & \overset{\text{ét-loc}}{\dashleftarrow\dashrightarrow} & (\widetilde{X}^+)^{\mathrm{Alb}} \\ \downarrow f^+ & & \downarrow \widetilde{f}^+ & \swarrow (\widetilde{f}^+)^{\mathrm{Alb}} & \\ Y^+ & \xleftarrow[/G]{} & \widetilde{Y}^+ & & \end{array}$$

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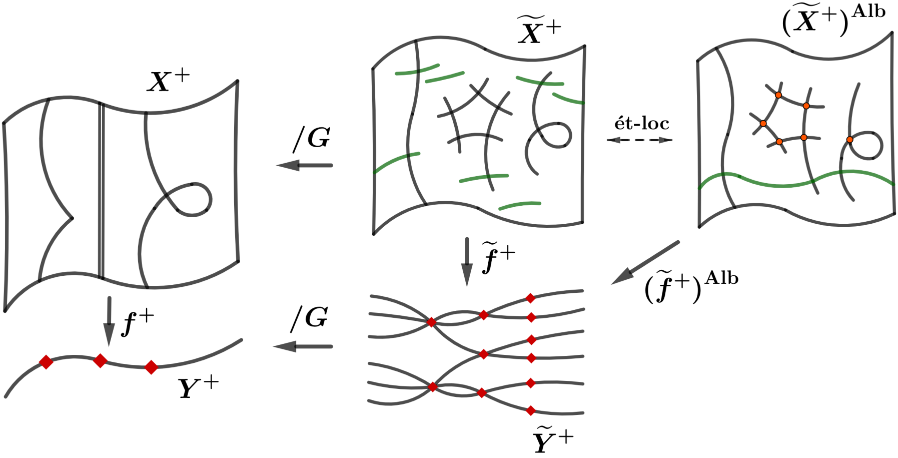

The sections of $(\widetilde{f}^+)^{\mathrm{Alb}}$ form a group scheme $P \to \widetilde{Y}^+$.

Proof: Step 4 (bounding the Tate–Shafarevich group)

Two important exact sequences: The component sequence

$$0 \to P^0 \to P \to \mu \to 0$$

where $\mu \to \widetilde{Y}^+$ is the relative component group of the Kulikov model, and the exponential exact sequence

$$0 \to \Gamma \to \mathfrak{p} \xrightarrow{\exp} P^0 \to 0$$

associated to the sheaf $\mathfrak{p}$ of Lie algebras of P. Note: Γ is a constructible sheaf of finitely generated $\mathbb{Z}$-modules.

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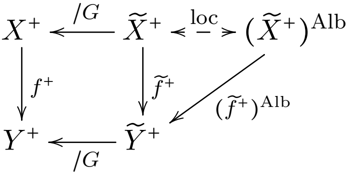
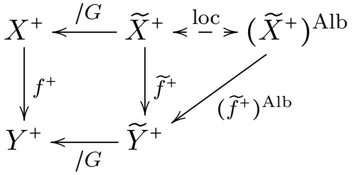


Multiple fibers of $f : X \to Y$ in codimension 1 are encoded by a *multiplicity class* $m(f) \in H^0(Y^+, \mathcal{H}^1(G, P))$. If two fibrations f, f' with equal Albanese have $m(f) = m(f')$, then the difference between their birational classes is measured by the G-equivariant sheaf cohomology group

$$t(f) - t(f') \in Ш_{G,\text{ét}} := H^1_G(\widetilde{Y}^+, P).$$

Thus, we are reduced to proving finiteness of $Ш_{G,\text{ét}}$.



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We must analyze the diagram

$$\begin{array}{ccccccc} & & & & H^1_{G,\,\text{ét}}(\widetilde{Y}^+, P^0) & & \\ & & & & \big\downarrow{\scriptstyle \text{analytification}} & & \\ H^1_G(\widetilde{Y}^+,\Gamma) & \longrightarrow & H^1_{G,\,\text{an}}(\widetilde{Y}^+,\mathfrak{p}) & \longrightarrow & H^1_{G,\,\text{an}}(\widetilde{Y}^+, P^0) & \longrightarrow & H^2_G(\widetilde{Y}^+,\Gamma) \end{array}$$

coupled with a general theorem of Raynaud and basic group cohomology, that the upper group is torsion. [n.b. torsion ≠ finite, cf. $\mathbb{Q}/\mathbb{Z}$]

We show the image of $H^1_G(\widetilde{Y}^+,\Gamma_{\mathbb{C}}) \to H^1_{G,\,\text{an}}(\widetilde{Y}^+,\mathfrak{p})$ receives a surjection from $H^2(X,\mathcal{O})$. Torsion-ness of $Ш_{G,\,\text{ét}}$ allows us to control the image of the analytification map.

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Philosophical point: Once we control the Tate–Shafarevich twist on a big open set $Y^+ \subset Y$, we can apply Hartogs' type results to prove that the analytification is injective.

This is far from true when $Y^+ \subset Y$ is not big.

Example

Consider an elliptic surface $S \to C$ and the result of logarithmic transforms $S' \to C$ at some points $p_i \in C$. These are biholomorphic over $C \setminus \{p_i\}$ but not bimeromorphic over any neighborhood of p_i.

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Theorem (Toy Theorem)

Let X, X' be two projective varieties. A bimeromorphism $\varphi : U \dashrightarrow U'$ on big open sets $U \subset X$, $U' \subset X'$ extends to a bimeromorphism $X \dashrightarrow X'$.

Takeaway: Using such Hartogs'-type results, it suffices to understand Tate–Shafarevich twists over a big open subset $Y^+ \subset Y$ of the base.

Proof: Step 4 (bounding the Tate–Shafarevich group)

Theorem (E.-Filipazzi-Greer-Mauri-Svaldi)

If $f : X \to Y$ is an abelian fibration of a K-trivial variety, then so is $f^{\mathrm{Alb}} : X^{\mathrm{Alb}} \to Y$. Similarly, if f is Lagrangian fibration of a primitive symplectic variety, then so is f^{Alb}.

Question

Can we construct new deformation classes of symplectic varieties, by passing to the Albanese of a Lagrangian fibration with multiple fibers?

Thank you for your time!

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Thank you for your time!