Stable degeneration of families of klt singularities with constant local volume

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Local volumes

Let $x \in (X, \Delta)$ be a klt singularity of dimension n, and v be a real valuation centered at x.

Definition (Ein–Lazarsfeld–Smith, 03)

The *volume* of v is

$$\mathrm{vol}_X(v) = \mathrm{mult}_X(\mathfrak{a}_\bullet(v)) = \lim_{\lambda \to \infty} \frac{\mathrm{length}(\mathscr{O}_{X,x}/\mathfrak{a}_\lambda(v))}{\lambda^n / n!}$$

where $\mathfrak{a}_\lambda(v) = \{h \in \mathscr{O}_{X,x} : v(h) \geq \lambda\}$ is the graded sequence of ideals (a.k.a., filtration) associated with v.

Local volumes

Definition (C. Li, 18)

The *normalized volume* of v is

$$\widehat{\mathrm{vol}}_{X,\Delta}(v) = A_{X,\Delta}(v)^n \cdot \mathrm{vol}_X(v).$$

It is "normalized" to be invariant under scaling of v. The *local volume* of $x \in (X, \Delta)$ is $\widehat{\mathrm{vol}}(x; X, \Delta) = \inf_v \widehat{\mathrm{vol}}_{X,\Delta}(v)$.

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Lemma (Y. Liu, 18)

$$\widehat{\mathrm{vol}}(x; X, \Delta) = \inf_{\mathfrak{a}_\bullet} \mathrm{lct}(X, \Delta; \mathfrak{a}_\bullet)^n \cdot \mathrm{mult}(\mathfrak{a}_\bullet).$$

where $\mathfrak{a}_\bullet$ *ranges over* $\mathfrak{m}_x$*-primary ideal filtrations of* $\mathscr{O}_{X,x}$.

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where $\mathfrak{a}_\bullet$ ranges over $\mathfrak{m}_x$-primary ideal filtrations of $\mathscr{O}_{X,x}$.

Note that if v^{m} is a minimizing valuation of $\widehat{\mathrm{vol}}_{X,\Delta}$, then $\mathfrak{a}_\bullet(v^{\mathrm{m}})$ is a minimizing filtration.

Stable degeneration

Theorem (conjectured by Li, 18; Li–Xu, 18)

(1) (Blum, 18) There exists a minimizer v^{m} *of the normalized volume function, that is,* $\widehat{\mathrm{vol}}(x; X, \Delta) = \widehat{\mathrm{vol}}_{X,\Delta}(v^{\mathrm{m}})$.

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(4) (Xu–Zhuang, 22) The graded ring $R_0 = \mathrm{gr}_{v^{\mathrm{m}}}(\mathscr{O}_{X,x})$ *is finitely generated.*

(5) (Li–Xu, 18) The induced degeneration $(X_0 = \mathrm{Spec}(R_0), \Delta_0)$ *is a K-semistable log Fano cone singularity (in particular, the pair* (X_0, Δ_0) *is klt).*

Stable degneration in families

Let S be a semi-normal scheme (essentially of finite type over a field of characteristic 0), and $\pi\colon (X, \Delta) \to S$ be a locally stable family of klt pairs, with a section $x\colon S \to X$.

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Theorem

Assume that the function $s \mapsto \widehat{\mathrm{vol}}(x_s; X_s, \Delta_s)$ is locally constant on S. Let v_s^{m} be the minimizer of $\widehat{\mathrm{vol}}$ for $x_s \in (X_s, \Delta_s)$, scaled such that $A_{X_s,\Delta_s}(v_s^{\mathrm{m}}) = 1$. Then there exist an ideal filtration $\mathfrak{a}_\bullet$ of $\mathscr{O}_X$ such that the following hold:

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1. $\mathfrak{a}_{s,\bullet} = \mathfrak{a}_\bullet(v_s^{\mathrm{m}})$ *for all* $s \in S$.
2. $\mathfrak{a}_\lambda/\mathfrak{a}_{>\lambda}$ *is supported on* $x(S)$ *and flat over* S *for all* $\lambda > 0$.
3. *The induced degeneration* $(X_0 = \mathrm{Spec}_S \bigoplus_\lambda \mathfrak{a}_\lambda/\mathfrak{a}_{>\lambda}, \Delta_0)$ *is a locally stable family of K-semistable log Fano cone singularities.*

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Remark (Blum–Liu, 21; Xu, 20)

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Therefore, for a general family of klt singularities, there is a stratification given by the local volumes, and the stable degeneration in families holds on each stratum (after semi-normalization).

Constant local volume

In the following, assume that S is the spectrum of a DVR, with the generic point $\eta \in S$, and the closed point $s \in S$. In this case, local stability means $(X, \Delta + X_s)$ is plt.

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Let $\mathfrak{a}_{\eta,\bullet} = \mathfrak{a}_\bullet(v_\eta^{\mathrm{m}}) \subseteq \mathscr{O}_{X_\eta}$. Since S is a DVR, we can extend $\mathfrak{a}_{\eta,\bullet}$ to a filtration $\mathfrak{a}_\bullet \subseteq \mathscr{O}_X$ such that each $\mathscr{O}_X/\mathfrak{a}_\lambda$ is flat over S and (set-theoretically) supported on the section $x(S)$.

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Then

$$\mathrm{length}(\mathscr{O}_{X_\eta, x_\eta}/\mathfrak{a}_{\eta,\lambda}) = \mathrm{length}(\mathscr{O}_{X_s, x_s}/\mathfrak{a}_{s,\lambda})$$

for all λ, so

$$\mathrm{mult}(\mathfrak{a}_{\eta,\bullet}) = \mathrm{mult}(\mathfrak{a}_{s,\bullet}).$$

Constant local volume

By the lower semi-continuity of log canonical thresholds,

$$\mathrm{lct}(X_s, \Delta_s; \mathfrak{a}_{s,\bullet}) \leq \mathrm{lct}(X_\eta, \Delta_\eta; \mathfrak{a}_{\eta,\bullet}).$$

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Then

$$\operatorname{lct}(X_s, \Delta_s; \mathfrak{a}_{s,\bullet})^n \cdot \operatorname{mult}(\mathfrak{a}_{s,\bullet}) \leq \operatorname{lct}(X_\eta, \Delta_\eta; \mathfrak{a}_{\eta,\bullet})^n \cdot \operatorname{mult}(\mathfrak{a}_{\eta,\bullet})$$

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Then

$$\begin{aligned}\mathrm{lct}(X_s, \Delta_s; \mathfrak{a}_{s,\bullet})^n \cdot \mathrm{mult}(\mathfrak{a}_{s,\bullet}) &\leq \mathrm{lct}(X_\eta, \Delta_\eta; \mathfrak{a}_{\eta,\bullet})^n \cdot \mathrm{mult}(\mathfrak{a}_{\eta,\bullet}) \\ &= \widehat{\mathrm{vol}}(x_\eta; X_\eta, \Delta_\eta) \\ &= \widehat{\mathrm{vol}}(x_s; X_s, \Delta_s).\end{aligned}$$

Thus $\mathfrak{a}_{s,\bullet}$ is a minimizing filtration.

Minimizing filtration

Theorem (Blum–Liu–Qi, 24)

A minimizing filtration is unique up to saturation and scaling.

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(1) $\mathfrak{a}_{s,\bullet} \subseteq \mathfrak{a}_\bullet(v_s^{\mathrm{m}})$,

(2) $\mathrm{mult}(\mathfrak{a}_{s,\bullet}) = \mathrm{mult}(\mathfrak{a}_\bullet(v_s^{\mathrm{m}}))$, and

(3) $w(\mathfrak{a}_{s,\bullet}) = w(\mathfrak{a}_\bullet(v_s^{\mathrm{m}}))$ for every real valuation w centered at $x_s \in X_s$ with positive volume (in particular, divisorial and quasi-monomial valuations).

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However, we want $\mathfrak{a}_{s,\bullet} = \mathfrak{a}_\bullet(v_s^{\mathrm{m}})$.

Divisorial minimizer

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Using that $\mathrm{lct}(X_s, \Delta_s; \mathfrak{a}_{s,\bullet}) = \mathrm{lct}(X_\eta, \Delta_\eta; \mathfrak{a}_{\eta,\bullet})$, we can get that

$$(Y, f_*^{-1}\Delta + E + Y_s)$$

is lc, that is, $(Y, f_*^{-1}\Delta + E) \to S$ is a locally stable family.

Divisorial minimizer

Let E'_s be a component of E_s. Then

$$A_{X_s,\Delta_s}(E'_s) = A_{X_\eta,\Delta_\eta}(E_\eta),$$

and

$$\mathrm{vol}(\mathrm{ord}_{E'_s}) \leq \mathrm{vol}(\mathrm{ord}_{E_s}).$$

Since the family has constant local volume, we conclude that E'_s induces the minimizing valuation v_s^{m}.

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By the uniqueness, $E'_s = E_s$, and it is a Kollár components for (X_s, Δ_s).

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Then $f_*\mathscr{O}_Y(-\lambda E)$ commutes with base change, so they form an ideal filtration with flat graded pieces, and coincide with the filtration of minimizing valuation on each fiber. This is the desired filtration in the Theorem.
Note that it coincide with $\mathfrak{a}_\bullet$ (up to scaling).

Higher rank

(1) Higher rank quasi-monomial minimizers lives on Kollár models (or models of dlt Fano type):

$$f\colon (Y, E = E_1 + \cdots + E_r) \to (X, \Delta)$$

where $(Y, f_*^{-1}\Delta + E)$ is dlt and $-(K_Y + f_*^{-1}\Delta + E)$ is ample, such that $v^{\mathrm{m}} \in \mathrm{QM}(Y, E)$.

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(3) But we need further modification (shrink the simplex $\mathrm{QM}(Y_\eta, E_\eta)$) to conclude the closed fiber is also dlt, hence get a locally stable family of Kollár models.