Open problems in K-stability

[For Fano varieties : K-stability $\overset{\text{YTD}}{\longleftrightarrow}$ Kähler-Einstein metric

— varieties w/ polarization:

expectation: " K-stability " $\longleftrightarrow$ cscK metric

Q1 Suppose $(Y, \Delta) \xrightarrow{\pi} (X, D)$ finite crepant morphism

b/t log Fano pairs ($\Leftrightarrow \pi^*(K_X+D) = K_Y+\Delta$)

Then: (Y, Δ) K-(semi/poly) stable

$\Leftrightarrow (X, D)$ K-(——) stable

This follows from ($\Rightarrow$) defn & ($\Leftarrow$) YTD (Liu-Zhu).

But is there an algebraic pf ?

(Rmk if π is Galois , known)

Q2 X Fano var. K-(semi/poly) stable

$\Rightarrow T_X$ slope (semi/poly) stable

(by YTD b/c KE metric is also a Hermitian-Einstein metric on T_X)

But is there an algebraic pf ?

(Rmk: same question for CY / general type)

Q3 Is every Fano variety birational to a Fano fibration w/ K-semistable general fibers ?

(Known: $\forall$ klt Fano var. has an isotrivial degen. to one that has Kähler-Ricci soliton)

Blum-Liu-Xu-Z.

§2 KLT singularities. $x \in X$

- local volume of KLT sing: $\widehat{\mathrm{vol}}(x, X) > 0$ (C.Li)

(· $C(V, -K_V) \Leftrightarrow \widehat{\mathrm{vol}} = (-K_V)^{n-1}$

V ↑ K-ss Fano

• in general: $\widehat{\mathrm{vol}} \longleftrightarrow$ volume density of KE)

• $x \in X$ klt $\Rightarrow$ $x_p \in X_p$ reduction mod p.

(Hara - Watanabe) strongly F-regular for $p \gg 0$

F-signature : $x \in X$ sing. in char p $(k = \bar{k})$

F = Frob.

$$F^e_* \mathcal{O}_X = \mathcal{O}_X^{a_e} \oplus N$$

no free summand.

$$s(x, X) = \lim_{e \to \infty} \frac{a_e}{p^{e \cdot \dim X}}, \quad \text{strongly F-reg} \Leftrightarrow s(x,X) > 0.$$

Q Is the limit $\liminf_{p \to \infty} s(x_p, X_p) > 0$?

Q4 Is it true that $\liminf_{p \to \infty} s(x_p, X_p) \geq C(\dim X) \cdot \widehat{\mathrm{vol}}(x, X)$

Property: • $y \in (Y, \Delta) \xrightarrow{\pi} x \in (X, D)$

finite cover of klt.

• if π is Galois $\xRightarrow{XZ}$ $\widehat{\mathrm{vol}}(y, Y, \Delta) = \deg(\pi) \cdot \widehat{\mathrm{vol}}(x, X, D)$

• any π $\xRightarrow{CR-S-T}$ $s(y, Y, \Delta) = \deg(\pi) \cdot s(x, X, D)$

Q5 Is this also true for non-Galois finite covers ?

Other properties of local volume

Fix $\varepsilon > 0$ & $n = \dim X$

• klt singularities of dim n & $\widehat{\mathrm{vol}} \geq \varepsilon$ are bounded up to isotrivial degeneration (stable degeneration)

• $\exists\, M = M(n, \varepsilon) > 0$ s.t.

$\forall$ klt singularities $x \in X$ of dim n & $\widehat{\mathrm{vol}} \geq \varepsilon$

$\Rightarrow$ embed. dim $(x, X) \leq M$, $\mathrm{mult}_x X \leq M$, ...

Q6 Fix ε, n. $\exists ?\ M = M(n, \varepsilon) > 0$ s.t. $\forall\, x \in X$ s.t. $s(x, X) \geq \varepsilon$

$\Rightarrow$ emb. dim $\leq M$, $\mathrm{mult}_x X \leq M$? ....

§3. CY moduli

Q7 Is there a proj. moduli space of (equivalence classes of) CY varieties ?

(Blum - Liu : dim 2 , boundary pol. CY)

(related to b - semiampleness conj.)

§4. General polarization (cscK)

Q8 cscK metric $\exists$ $\Longleftrightarrow$ stability condition.

(which one ?)

uniform K-stable (origin) $\Longleftarrow$ cscK + Aut $< \infty$

$\Updownarrow$ divisorial stability (Boucksom - Jonsson)

$\Updownarrow$

uniform K-stab. for model (Chi Li) $\Longrightarrow$ cscK + Aut $< \infty$

Q9 - ?

- Is K-stability an open condition in family (X, L) or in ample cone ?
- (X_i, L_i) K-stable $\Rightarrow$ is product $(X_1 \times X_2, L_1 \boxtimes L_2)$ K-stable ?
- $\mathrm{Aut}(X, L)$ reductive when (X, L) K-polystable ?